Mathematics 204

Summer 2013

Exam III

Your Printed Name: Solution

Your Instructor's Name: ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 6 pages of problems containing 6 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. Express all solutions in real-valued, simplified form.
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	20	20	20	20	20	100

1.[20]Find the solution $y = y(t)$ of the initial value problem $y'' + y' - 2y = -3\delta(t-1)$, $y(0) = 0$, $y'(0) = 3$. Write your final answer for the solution in terms of a piecewise defined function.

$$\mathcal{L}\{y'' + y' - 2y\} = \mathcal{L}\{-3\delta(t-1)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = -3\mathcal{L}\{\delta(t-1)\}$$

$$s^2\mathcal{L}\{y\} - s\overset{0}{y(0)} - \overset{3}{y'(0)} + s\mathcal{L}\{y\} - \overset{0}{y(0)} - 2\mathcal{L}\{y\} = -3e^{-1 \cdot s}$$

$$(s^2 + s - 2)\mathcal{L}\{y\} - 3 = -3e^{-s}$$

$$\mathcal{L}\{y\} = \frac{3}{s^2+s-2} - \frac{3e^{-s}}{s^2+s-2}$$

$\mathcal{L}.F.D.$

$$\frac{3}{s^2+s-2} = \frac{3}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} =$$

$$= \frac{As - A + Bs + 2B}{(s+2)(s-1)}$$

$A = -B$

$\Rightarrow 3 = -A + 2(-A) = -3A$

$A = -1 \Rightarrow B = 1$

$$y = \mathcal{L}^{-1}\left\{\frac{-1}{s+2} + \frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{-1}{s+2} + \frac{1}{s-1}\right)\right\}$$

$$y = -e^{-2t} + e^{t} - u_1(t)\, f(t-1) \quad \text{with } f(t) = -e^{-2t} + e^{t}$$

$$y = -e^{-2t} + e^{t} - u_1(t) \cdot \left[-e^{-2(t-1)} + e^{(t-1)}\right]$$

$$y(t) = \begin{cases} -e^{-2t} + e^{t} & 0 \le t < 1 \\ -e^{-2t} + e^{t} + e^{-2(t-1)} - e^{(t-1)} & t \geq 1 \end{cases}$$

2. a) [20] Find the solution $y = y(t)$ of the initial value problem $y''(t) + y(t) = u_\pi(t)$, $y(0) = 1$, $y'(0) = 0$.

b) [ ] Is $y(\pi/2)$ greater than $y(3\pi/2)$?

a) $\mathcal{L}\{y'' + y\} = \mathcal{L}\{u_\pi(t)\}$

$$s^2\mathcal{L}\{y\} - s\overset{1}{y(0)} - \overset{0}{y'(0)} + \mathcal{L}\{y\} = \frac{e^{-\pi s}}{s}$$

$$\mathcal{L}\{y\}(s^2+1) = s + \frac{e^{-\pi s}}{s}$$

$$\mathcal{L}\{y\} = \frac{s}{s^2+1} + e^{-\pi s}\cdot\frac{1}{s(s^2+1)}$$

P.F.D.

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2 + A + Bs^2 + Cs}{s(s^2+1)}$$

$\Rightarrow C = 0,\ A = 1,\ B = -1$

$$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{s(s^2+1)}\right\}$$

$c = \pi$

$$y = \cos t + u_\pi(t)\cdot f(t-\pi)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-s}{s^2+1}\right\} = \underline{1 - \cos t}$

$$y = \cos t + u_\pi(t)\left(1 - \cos(t-\pi)\right)$$

b) $y(\pi/2) = \cos(\pi/2) + \overset{\pi/2<\pi}{0}\cdot\left(1 - \cos(\pi/2 - \pi)\right) = \cos(\pi/2) = \underline{\underline{0}}$

$y(\tfrac{3}{2}\pi) = \cos(\tfrac{3}{2}\pi) + \underset{\frac{3}{2}\pi>\pi}{1}\cdot\left(1 - \cos(\tfrac{3}{2}\pi - \pi)\right) =$

$$= 0 + 1 - \underbrace{\cos(\tfrac{1}{2}\pi)}_{=0} = \underline{\underline{1}}$$

$\Rightarrow y(\tfrac{3}{2}\pi) > y(\pi/2)$

3.[20] Solve the integral equation $y(t)=\int_0^t e^{t-\tau}y(\tau)\,d\tau+2$.

Note: $\boxed{\int_0^t e^{t-\tau}y(\tau)\,d\tau = e^t * y(t)}$

$$\mathcal{L}\{y\} = \mathcal{L}\{\int_0^t e^{t-\tau}y(\tau)\,d\tau\} + \mathcal{L}\{2\}$$

$$\mathcal{L}\{y\} = \mathcal{L}\{e^t * y(t)\} + \frac{2}{s}$$

$$\mathcal{L}\{y\} = \mathcal{L}\{e^t\}\cdot\mathcal{L}\{y\} + \frac{2}{s}$$

$$\mathcal{L}\{y\} = \frac{1}{s-1}\cdot\mathcal{L}\{y\} + \frac{2}{s}$$

$$\mathcal{L}\{y\}\left[1-\frac{1}{s-1}\right] = \frac{2}{s}$$

$$\mathcal{L}\{y\}\left[\frac{s-1-1}{s-1}\right] = \frac{2}{s}$$

$$\mathcal{L}\{y\}\left[\frac{s-2}{s-1}\right] = \frac{2}{s} \Rightarrow \mathcal{L}\{y\} = \frac{2(s-1)}{s\cdot(s-2)} = \frac{2\cancel{s}}{\cancel{s}(s-2)} - \frac{2}{s(s-2)}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left\{\frac{2}{s-2} - \frac{2}{s(s-2)}\right\} = 2e^{2t} - \mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{1}{s-2}\right\} \Rightarrow$$

P.F.D.

$$\frac{2}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} = \frac{As-2A+Bs}{s(s-2)} \Rightarrow A=-1,\ B=1$$

$$y = 2e^{2t} + 1 - e^{2t} = \underline{e^{2t}+1}$$

4. [20] Find general solution to $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$.

1) Get λ $\quad \det(A - \lambda \cdot I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)\cdot(-2-\lambda) - 4 = -2 - \lambda + 2\lambda + \lambda^2 - 4 =$$

$$= \lambda^2 + \lambda - 6 = 0$$

$$\lambda_{1/2} = \frac{-1 \pm \sqrt{1 - 4\cdot(-6)}}{2} = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

$$\lambda_1 = \frac{-1+5}{2} = \frac{4}{2} = 2 \qquad \lambda_2 = \frac{-1-5}{2} = \frac{-6}{2} = -3$$

2) Get eigenvectors

i) $\lambda_1 = 2 \quad (A - \lambda_1 I)\vec{v}^{(1)} = 0 \Rightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right) \xrightarrow{\text{Gauß}} \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Pick $v_2 = c \rightarrow -1 \cdot v_1 + 2v_2 = 0 \quad v_1 = 2v_2$

$\Rightarrow$ Pick $v_2 = 1 \rightarrow v_1 = 2 \cdot v_2 = 2 \quad \Rightarrow \vec{v}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

ii) $\lambda_2 = -3 \quad (A - \lambda_2 I)\vec{v}^{(2)} = 0 \Rightarrow \left(\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{\text{Gauß}} \left(\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ Pick $v_2 = c \rightarrow 4 \cdot v_1 + 2v_2 = 0 \rightarrow v_1 = -\frac{2}{4} v_2 = -\frac{1}{2} v_2$

Pick $v_2 = 2 \rightarrow v_1 = -1 \quad \Rightarrow \vec{v}^{(2)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

3) Solutions are

$$\vec{x}^{(1)} = \vec{v}^{(1)} e^{\lambda_1 t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} \ \& \ \vec{x}^{(2)} = \vec{v}^{(2)} e^{\lambda_2 t} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-3t}$$

$$\Rightarrow \text{m.g.s.} \ \boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-3t}}$$

5. [20] Transform the given second order initial value problem into an equivalent system of first order differential equations with initial conditions. Express your final answer using vector-matrix notation."

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(0) = a, y'(0) = b.$$

$\boxed{\text{Define } x_1 = y \quad x_2 = y'}$ $\Rightarrow$ $x_1' = y' = x_2$

$$x_2' = y'' = g(t) - p(t)y' - q(t)y \overset{\text{change order}}{=} -q(t)y - p(t)y' + g(t) \overset{\text{express i.t.o. } \underline{x}}{=} -q(t)x_1 - p(t)x_2 + g(t)$$

$$\Rightarrow \quad \vec{x}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

2013 Summer Semester, Math 204 Hour Exam _3___
Instructor __Sabrina Streipert___, Section ___A__

100	4	59	1	19	
99		58		18	
98	3	57	1	17	
97		56	1	16	
96	3	55	1	15	
95	2	54		14	
94	2	53		13	
93	1	52		12	
92	1	51	1	11	
91		50		10	
90	1	49		9	1
89		48	2	8	
88		47		7	
87	1	46	2	6	
86	1	45		5	
85	2	44	1	4	
84		43	1	3	
83		42		2	
82	2	41		1	
81	2	40		0	
80		39			
79		38			
78		37			
77		36	1		
76		35			
75	2	34			
74	1	33	2		
73		32			
72		31			
71		30			
70		29	1		
69		28			
68		27			
67		26			
66	1	25			
65	1	24	1		
64		23			
63		22			
62	3	21			
61		20	1		
60	2				

Number taking exam:_53__
Median: __75.0___
Mean: __69.8____
Standard Deviation: 24.9___

Number receiving A's:____17___
Number receiving B's:____8__
Number receiving C's:_____3__
Number receiving D's:____7___
Number receiving F's:____18___