Mathematics 204

Fall 2013

Final Exam

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. The final exam consists of this cover page, 7 pages of problems containing 7 numbered problems, and a short table of Laplace transform formulas.
5. Once the exam begins, you will have 120 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, work must be shown for all integration, partial fraction, and matrix computations.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 200.

problem	1	2	3	4	5	6	7	**Sum**
points earned								
maximum points	36	26	36	25	25	26	26	200

1. Find the general solution of each of the following differential equations.

(a) [18] $ty' + 2y = \sin(t)$ (First order, linear)

$$y' + \frac{2}{t}y = \frac{\sin(t)}{t}$$

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln(t) + \cancel{c}^{\,0}} = e^{\ln(t^2)} = t^2$$

$$t^2\left(y' + \frac{2}{t}y\right) = t^2\left(\frac{\sin(t)}{t}\right)$$

$$\underbrace{t^2y' + 2ty}_{\text{Exact!}} = t\sin(t)$$

$$\frac{d}{dt}\left[t^2 y\right] = t\sin(t)$$

$$\therefore\ t^2 y = \int \overset{u}{t}\,\overbrace{\sin(t)dt}^{dv} = -t\cos(t) - \int -\cos(t)dt = -t\cos(t) + \sin(t) + c$$

General Solution:

$$\boxed{y(t) = \frac{-t\cos(t) + \sin(t) + c}{t^2}}$$

(b) [18] $ty' - y = y^2$

$$\left(\frac{dy}{dt} =\right) y' = \frac{y^2 + y}{t} \quad \text{(First order, separable)}$$

$$\frac{dy}{y(y+1)} = \frac{dt}{t}$$

From PFD at right

$$\int \left(\overbrace{\frac{1}{y} - \frac{1}{y+1}}\right)dy = \int \frac{1}{t}\,dt$$

$$\ln|y| - \ln|y+1| = c + \ln|t|$$

$$\ln\left|\frac{y}{y+1}\right| = c + \ln|t|$$

$$\left|\frac{y}{y+1}\right| = e^{c + \ln|t|} = \underbrace{e^c}_{A} \cdot e^{\ln|t|}$$

$$\frac{y}{y+1} = \pm At$$

Partial Fraction Decomposition:

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$\Rightarrow \quad 1 = A(y+1) + By$$

Set $y = 0$ to find A: $\quad 1 = A$

Set $y = -1$ to find B: $\quad 1 = -B$

$$\therefore \frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

$$y = \pm At(y+1)$$

$$y \mp Aty = \pm At$$

$$y(1 \mp At) = \pm At$$

$$\therefore \boxed{y(t) = \frac{At}{1 - At}}$$

where A is an arbitrary constant.

2.[26] Solve the initial value problem $y'' + 2y' + y = e^{-t}$, $y(0) = 1$, $y'(0) = -2$.

(Second order, linear, constant coefficients, nonhomogeneous)

$y = e^{rt}$ in $y'' + 2y' + y = 0$ leads to $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$

$\Rightarrow r = -1$ (multiplicity two). Therefore $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$.

To find a particular solution, we use variation of parameters:

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

where $u_1(t) = \int -\frac{g(t) y_2(t)}{W(t)}\,dt$ and $u_2(t) = \int \frac{g(t) y_1(t)}{W(t)}\,dt$ with

$g(t) = e^{-t}$, $y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$, and $W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix}$

$= (1-t)e^{-2t} + te^{-2t} = e^{-2t}$. Therefore

$$u_1(t) = \int \frac{-e^{-t}(te^{-t})}{e^{-2t}}\,dt = \int -t\,dt = -\frac{t^2}{2} + \cancelto{0}{c},$$

$$u_2(t) = \int \frac{e^{-t} \cdot e^{-t}}{e^{-2t}}\,dt = \int 1\,dt = t + \cancelto{0}{c},$$

so $y_p(t) = \left(-\frac{t^2}{2}\right)e^{-t} + (t)\,te^{-t} = \frac{t^2}{2} e^{-t}$.

The general solution is $y(t) = y_c(t) + y_p(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{t^2}{2} e^{-t}$. Thus

$1 = y(0) = c_1$ and $-2 = y'(0) = -c_1 + c_2 \Rightarrow c_2 = -1$.

$$\therefore \boxed{y(t) = e^{-t} - te^{-t} + \frac{t^2}{2} e^{-t}}$$

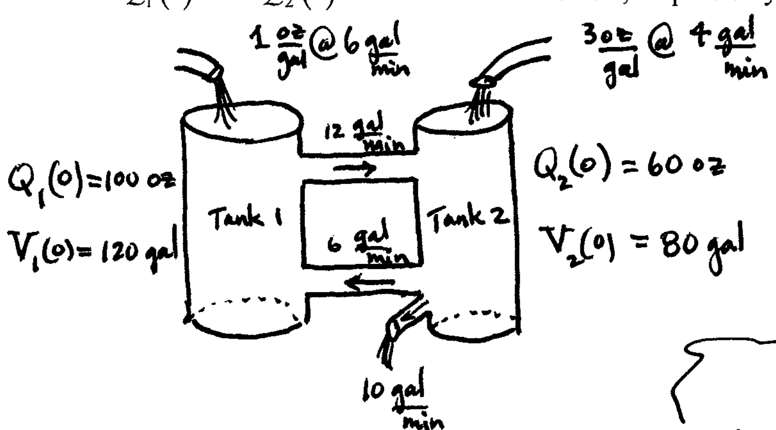
3. (a) [18] A series circuit has a capacitor of 0.25×10^{-6} farads, a resistor of 5×10^{3} ohms, and an inductor of 1 henry. The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at $t=0$, write **BUT DO NOT SOLVE** an initial value problem that models the charge $Q(t)$ on the capacitor at all times $t\ge 0$.

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

where $L = 1\,H$, $R = 5\times10^{3}\,\Omega$, $C = \frac{1}{4}\times10^{-6}\,F$ and $E(t) = 12\,V$.

$$\boxed{Q'' + 5000Q' + 4{,}000{,}000Q = 12, \quad Q(0)=0, \quad Q'(0)=0}$$

(b) [18] Consider two interconnected tanks. Tank 1 initially contains 120 gallons of water and 100 ounces of salt, and Tank 2 initially contains 80 gallons of water and 60 ounces of salt. Water containing 1 ounce per gallon of salt flows into Tank 1 at a rate of 6 gallons per minute. The mixture flows from Tank 1 into Tank 2 at a rate of 12 gallons per minute. Water containing 3 ounces per gallon of salt also flows into Tank 2 at a rate of 4 gallons per minute (from the outside). The mixture drains from Tank 2 at a rate of 16 gallons per minute, of which some flows into Tank 1 at a rate of 6 gallons per minute, while the remainder leaves the system. Write **BUT DO NOT SOLVE** differential equations and initial conditions that model the number of ounces $Q_1(t)$ and $Q_2(t)$ of salt in Tanks 1 and 2, respectively, at time $t\ge 0$.



Assuming the tanks are well-stirred and the net rate of change of salt per unit time in each tank is the rate at which salt flows into the tank minus the rate at which salt flows out of the tank, we have:

(Tank 1) $$Q_1'(t) = \left(6\,\frac{\text{gal}}{\text{min}}\right)\left(1\,\frac{\text{oz}}{\text{gal}}\right) + \left(6\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{Q_2(t)\,\text{oz}}{80\,\text{gal}}\right) - \left(12\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{Q_1(t)\,\text{oz}}{120\,\text{gal}}\right)$$

(Tank 2) $$Q_2'(t) = \left(4\,\frac{\text{gal}}{\text{min}}\right)\left(3\,\frac{\text{oz}}{\text{gal}}\right) + \left(12\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{Q_1(t)\,\text{oz}}{120\,\text{gal}}\right) - \left(16\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{Q_2(t)\,\text{oz}}{80\,\text{gal}}\right)$$

Note: The rate at which fluid flows into each tank is equal to the rate at which fluid flows out of that tank. Therefore the volume of fluid in each tank at any time is constant.

Simplifying yields

$$\boxed{\begin{aligned} Q_1' &= -\tfrac{1}{10}Q_1 + \tfrac{3}{40}Q_2 + 6, \quad Q_1(0)=100,\\ Q_2' &= \tfrac{1}{10}Q_1 - \tfrac{1}{5}Q_2 + 12, \quad Q_2(0)=60.\end{aligned}}$$

4. (a) [21] Solve the initial value problem $y''+2y'+2y=\delta(t-\pi)$, $y(0)=0$, $y'(0)=2$.
(b) [4] Evaluate the solution to part (a) at the two points $t=\pi/2$ and $t=3\pi/2$.

(a) We use the method of Laplace transforms because of the presence of the Dirac delta term.

$$\mathcal{L}\{y''+2y'+2y\}(s)=\mathcal{L}\{\delta(t-\pi)\}(s).$$

Applying linearity of the Laplace transform and formulas 6 and 9 in the Laplace transform table yield

$$s^2Y(s)-s\overset{0}{y(0)}-\overset{2}{y'(0)}+2(sY(s)-\overset{0}{y(0)})+2Y(s)=e^{-\pi s}.$$

Simplifying,

$$(s^2+2s+2)Y(s)=2+e^{-\pi s}$$

$$Y(s)=\frac{2}{s^2+2s+2}+\frac{1}{s^2+2s+2}\cdot e^{-\pi s}$$

so $$y(t)=\mathcal{L}^{-1}\{Y(s)\}=2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\cdot e^{-\pi s}\right\}.$$

But formulas 7 and 3 in the Laplace transform table give

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=e^{-t}\sin(t)$$

and formula 8 and the result above yield

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{(s+1)^2+1}\right\}=u_\pi(t)e^{-(t-\pi)}\sin(t-\pi).$$

Therefore $\boxed{y(t)=2e^{-t}\sin(t)+u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)}$ solves the IVP.

(b) $$y\left(\tfrac{\pi}{2}\right)=2e^{-\frac{\pi}{2}}\overset{1}{\sin\left(\tfrac{\pi}{2}\right)}+\overset{0}{u_\pi\left(\tfrac{\pi}{2}\right)}e^{\frac{\pi}{2}}\sin\left(-\tfrac{\pi}{2}\right)=\boxed{2e^{-\frac{\pi}{2}}}$$

$$y\left(\tfrac{3\pi}{2}\right)=2e^{-\frac{3\pi}{2}}\underbrace{\sin\left(\tfrac{3\pi}{2}\right)}_{-1}+\overset{1}{u_\pi\left(\tfrac{3\pi}{2}\right)}e^{-\frac{\pi}{2}}\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1}=\boxed{-2e^{-\frac{3\pi}{2}}+e^{-\frac{\pi}{2}}}$$

5.[25] Solve the initial value problem $y'(t)-\frac{1}{2}\int_0^t (t-\xi)^2 y(\xi)\,d\xi = -t,\ y(0)=1.$

We use the Laplace transform method because of the presence of the convolution term:

$$(f*y)(t) = \int_0^t (t-\xi)^2 y(\xi)\,d\xi \qquad \text{where} \quad f(t)=t^2.$$

$$\therefore\ \mathcal{L}\left\{ y'(t) - \tfrac{1}{2}(f*y)(t) \right\}(s) = \mathcal{L}\{-t\}(s).$$

Using linearity of the Laplace transform and formulas 6, 5, and 2 yield

$$sY(s) - \overset{1}{y(0)} - \frac{1}{2}F(s)Y(s) = -\frac{1}{s^2}$$

$$sY(s) - 1 - \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{s^3}\right)Y(s) = -\frac{1}{s^2}.$$

Simplify, we have

$$\left(s - \frac{1}{s^3}\right)Y(s) = 1 - \frac{1}{s^2}$$

$$\left(\frac{s^4-1}{s^3}\right)Y(s) = \frac{s^2-1}{s^2}$$

$$Y(s) = \frac{s^2-1}{s^2}\cdot\frac{s^3}{s^4-1} = \frac{s}{s^2+1}.$$

Therefore

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \boxed{\cos(t)}$$

by formula 4 in the Laplace transform table.

6. [26] Solve the system $u' = 2u + 8v$, $v' = -u - 2v$, subject to the initial conditions $u(0) = 2$, $v(0) = -1$.

Method I : Substitution.

Solving the second equation for u yields $u = -v' - 2v$. Substituting for u in the first equation gives $(-v'-2v)' = 2(-v'-2v) + 8v$. Simplifying yields $-v'' - 2v' = -2v' - 4v + 8v$ or equivalently $0 = v'' + 4v$. Then assuming $v(t) = e^{rt}$ for some constant r leads to $0 = r^2e^{rt} + 4e^{rt}$ or $0 = r^2 + 4$ so $r = \pm 2i$. Consequently $v(t) = \tilde{c}_1 e^{2it} + \tilde{c}_2 e^{-2it} = \tilde{c}_1(\cos(2t) + i\sin(2t)) + \tilde{c}_2(\cos(-2t) + i\sin(-2t))$ $= c_1\cos(2t) + c_2\sin(2t)$ where $c_1 = \tilde{c}_1 + \tilde{c}_2$ and $c_2 = i\tilde{c}_1 - i\tilde{c}_2$ are arbitrary constants.

Then $u(t) = -v'(t) - 2v(t) = -(-2c_1\sin(2t) + 2c_2\cos(2t)) - 2(c_1\cos(2t) + c_2\sin(2t)) =$ $(-2c_2 - 2c_1)\cos(2t) + (2c_1 - 2c_2)\sin(2t)$. Applying the initial conditions yields

$-1 = v(0) = c_1\cos(0) + c_2\sin(0) = c_1$ and $2 = u(0) = (-2c_1 - 2c_2)\cos(0) + (2c_1 - 2c_2)\sin(0)$

$\Rightarrow 2 = -2c_1 - 2c_2 = -2(-1) - 2c_2 = 2 - 2c_2$

so $c_2 = 0$.

Therefore $\boxed{u(t) = 2\cos(2t) - 2\sin(2t)}$ and $\boxed{v(t) = -\cos(2t)}$ solve the IVP.

Method II : Vector-Matrix.

Let $\vec{x} = \begin{bmatrix} u \\ v \end{bmatrix}$. Then $\vec{x}' = \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 2u + 8v \\ -u - 2v \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\vec{x}$ where $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$.

Substituting $\vec{x} = \vec{k}e^{rt}$ into $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$, the eigenvalue equation for the matrix A. We calculate the eigenvalues r and eigenvectors $\vec{k}$ for the matrix A as follows.

$$0 = \det(A - rI) = \begin{vmatrix} 2-r & 8 \\ -1 & -2-r \end{vmatrix} = (2-r)(-2-r) + 8 = r^2 + 4 \quad \text{so } r = \pm 2i.$$

If $r = 2i$ then the eigenvalue equation $(A - rI)\vec{k} = \vec{0}$ becomes

$$\begin{bmatrix} 2-2i & 8 \\ -1 & -2-2i \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(continued on next page)

This is equivalent to the system $\begin{cases} (2-2i)k_1 + 8k_2 = 0, \\ -k_1 - (2+2i)k_2 = 0. \end{cases}$

But the first equation is redundant since it is $-(2-2i)$ times the second equation. Therefore, to solve the system we must just solve the second eqn.:
$k_1 = -(2+2i)k_2$ where k_2 is arbitrary. Thus

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -(2+2i)k_2 \\ k_2 \end{bmatrix} = -k_2 \begin{bmatrix} 2+2i \\ -1 \end{bmatrix}.$$

Taking $k_2 = -1$ yields the following:

Eigenvalues of A	Eigenvectors of A
$r_1 = 2i$	$\vec{k}^{(1)} = \begin{bmatrix} 2+2i \\ -1 \end{bmatrix}$
$r_2 = -2i = \overline{r_1}$	$\vec{k}^{(2)} = \overline{\vec{k}^{(1)}} = \begin{bmatrix} 2-2i \\ -1 \end{bmatrix}$

Then $\vec{x}(t) = \vec{k}^{(1)} e^{r_1 t} = \begin{bmatrix} 2+2i \\ -1 \end{bmatrix} e^{2it}$ is a complex solution to $\vec{x}' = A\vec{x}$.

Note $\vec{x}(t) = \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)\left(\cos(2t) + i\sin(2t)\right) = \left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) + i\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(2t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\sin(2t)\right).$

The real part and the imaginary part of $\vec{x}$ form a real fundamental set of solutions to $\vec{x}' = A\vec{x}$. Therefore the general solution is

$$c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) = c_1\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) + c_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(2t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\sin(2t)\right).$$

Applying the initial conditions gives

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = c_1\vec{x}^{(1)}(0) + c_2\vec{x}^{(2)}(0) = c_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

By inspection, $c_1 = 1$ and $c_2 = 0$. Therefore the solution of the IVP is

$$\boxed{\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t) = \begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ -\cos(2t) \end{bmatrix}.}$$

7. [26] Let $A = \begin{pmatrix} \frac{-1}{10} & \frac{3}{40} \\ \frac{1}{10} & \frac{-1}{5} \end{pmatrix}$. Given that $\Psi(t) = \begin{pmatrix} 3e^{-t/20} & -e^{-t/4} \\ 2e^{-t/20} & 2e^{-t/4} \end{pmatrix}$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = A\mathbf{x}$, find the general solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 6 \\ 12 \end{pmatrix}$.

The general solution of the nonhomogeneous system is $\vec{x} = \vec{x}_c(t) + \vec{x}_p(t)$ where $\vec{x}_c(t) = \Psi(t)\vec{c}$ ($\vec{c}$ = arbitrary constant vector) is the general solution of the associated homogeneous system $\vec{x}' = A\vec{x}$ and $\vec{x}_p(t)$ is any particular solution to the nonhomogeneous system $\vec{x}' = A\vec{x} + \begin{bmatrix} 6 \\ 12 \end{bmatrix}$. It is most convenient to use the method of undetermined coefficients to find $\vec{x}_p$ in this problem. Since the driver term in the nonhomogeneous system is $\vec{g}(t) = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$, we expect a particular solution of the form $\vec{x}_p(t) = \vec{k}$ where $\vec{k}$ is a constant vector. Then

$$\vec{0} = \vec{k}' = \vec{x}_p' = A\vec{x}_p + \begin{bmatrix} 6 \\ 12 \end{bmatrix} = A\vec{k} + \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

so solving for $\vec{k}$ yields $\vec{k} = -A^{-1}\begin{bmatrix} 6 \\ 12 \end{bmatrix} = -\frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} 6 \\ 12 \end{bmatrix}$.

Note that $\det(A) = \frac{1}{50} - \frac{3}{400} = \frac{8-3}{400} = \frac{5}{400} = \frac{1}{80}$. Therefore

$$\vec{k} = -80\begin{bmatrix} -\frac{1}{5} & -\frac{3}{40} \\ -\frac{1}{10} & -\frac{1}{10} \end{bmatrix}\begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 16 & 6 \\ 8 & 8 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 168 \\ 144 \end{bmatrix}.$$ Consequently,

$$\boxed{\vec{x}(t) = \begin{bmatrix} 3e^{-t/20} & -e^{-t/4} \\ 2e^{-t/20} & 2e^{-t/4} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 168 \\ 144 \end{bmatrix}}$$

is the general solution of the nonhomogeneous system where c_1 and c_2 are arbitrary constants.

A SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\dfrac{1}{s-a}$
2. t^n	$\dfrac{n!}{s^{n+1}}, \quad n = 0, 1, 2, 3....$
3. $\sin(bt)$	$\dfrac{b}{s^2+b^2}$
4. $\cos(bt)$	$\dfrac{s}{s^2+b^2}$
5. $(f * g)(t)$	$F(s)G(s)$
6. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
7. $e^{ct} f(t)$	$F(s-c)$
8. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
9. $\delta(t-c)$	e^{-cs}