Math 3304 Fall 2016 Exam 1

Your printed name: Dr. Grow

Your instructor's name: ____________________

Your section (or Class Meeting Days and Time): ____________________

Instructions:

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e., not on vibrate) during the exam.
3. This exam is closed book and closed notes. No calculators or other electronic devices are allowed.
4. Exam 1 consists of this cover page, and 4 pages of problems containing 5 numbered problems.
5. Once the exam begins, you will have 50 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

Problem	1	2	3	4	5	Sum
Points Earned						
Max. Points	16	18	22	22	22	100

1. [16] Classify each differential equation by completing the columns in the following table.

Differential Equation	Order?	Linear? (Y/N)
$\cos(x)y + x^2 - y' = 0$	1	Y
$y'' - t^{-5} + (y')^3 = 0$	2	N
$y''' = y(y+1)$	3	N
$t\dfrac{d^2u}{dt^2} - e^t\dfrac{d^4u}{dt^4} = 0$	4	Y

2. [18] Consider, **BUT DO NOT SOLVE**, the differential equation $y' = y^2(4-y)(1+y)$.

a) Determine the equilibrium solutions (critical points) of the differential equation.
b) Classify each equilibrium solution as either asymptotically stable, unstable, or semi-stable. **Show your work.**
c) If $y(t)$ denotes the solution of the differential equation satisfying the initial condition $y(0) = 2$, determine $\lim_{t\to\infty} y(t)$.

(a) Equilibrium solutions of autonomous DEs are constant and hence have derivative zero. Thus they satisfy $0 = y^2(4-y)(1+y)$. Consequently $\boxed{y=0,\ y=4, \text{ and } y=-1}$ are the equilibrium solutions.

(b)

Interval	Sign of $y' = y^2(4-y)(1+y)$
$4 < y < \infty$	$(+)(-)(+) = -$
$0 < y < 4$	$(+)(+)(+) = +$
$-1 < y < 0$	$(+)(+)(+) = +$
$-\infty < y < -1$	$(+)(+)(-) = -$

Phaseline

$\boxed{4 = y \text{ is asymptotically stable}}$

$\boxed{0 = y \text{ is semi-stable}}$

$\boxed{-1 = y \text{ is unstable}}$

(c) If $y(0) = 2$ then the solution increases to the stable equilibrium solution $y = 4$. Therefore

$$\boxed{\lim_{t\to\infty} y(t) = 4}.$$

3. [22] Find the general solution $y(t)$ of the differential equation

$$t y' - 2y = t^4 \cos(t).$$

Linear, 1^{st} order.

Normalize the DE: $y' - \frac{2}{t} y = t^3 \cos(t)$

An integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{2}{t}dt} = e^{-2\ln|t| + \cancel{c}^{\,0}} = t^{-2}$.

Multiplying the normalized DE through by the integrating factor yields

$$t^{-2}\left[y' - 2t^{-1}y\right] = t^{-2}\left[t^3 \cos(t)\right]$$

$$t^{-2}y' - 2t^{-3}y = t\cos(t)$$

$$\frac{d}{dt}\left(t^{-2}y\right) = t\cos(t)$$

Integrating both sides gives

$$t^{-2}y = \int t\cos(t)\,dt$$

$u = t$, $dv = \cos(t)dt$

$du = dt$, $v = \sin(t)$

$$= t\sin(t) - \int \sin(t)\,dt$$

$$= t\sin(t) + \cos(t) + c$$

Therefore $\boxed{y(t) = t^3\sin(t) + t^2\cos(t) + ct^2}$ (c an arbitrary constant)

is the general solution.

4. [22] Find the explicit solution of the initial value problem

$$\sqrt{1+x^2}\,\frac{dy}{dx} = x\,y^3, \quad y(0)=1.$$

Nonlinear, 1st order

$$\frac{dy}{dx} = \underbrace{\frac{x}{\sqrt{1+x^2}}}_{g(x)} \cdot \underbrace{y^3}_{h(y)}$$

Therefore the equation is separable.

$$y^{-3}\,dy = \frac{x\,dx}{\sqrt{1+x^2}}$$

Now we integrate both sides.

$$\frac{y^{-2}}{-2} + c_1 = \int y^{-3}\,dy = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{1+x^2}}$$

Let $u = 1+x^2$. Then $du = 2x\,dx$.

$$-\frac{1}{2y^2} + c_1 = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + c_2$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c \qquad (c = c_2 - c_1)$$

$$\frac{-1}{2\left(c+\sqrt{1+x^2}\right)} = y^2$$

$$\pm\sqrt{\frac{-1}{2\left(c+\sqrt{1+x^2}\right)}} = y(x)$$

In order to satisfy the initial condition $y(0)=1$ (>0) we must choose the $+$ sign. Also, we must choose c such that

$$\frac{-1}{2\left(c+\sqrt{1+0^2}\right)} = 1, \qquad \text{i.e. } c+1 = -\frac{1}{2}.$$

Hence $c = -3/2$. Consequently,

$$y(x) = \sqrt{\frac{-1}{2\left(-\frac{3}{2}+\sqrt{1+x^2}\right)}} = \sqrt{\frac{-1}{-3+2\sqrt{1+x^2}}} = \boxed{\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}}$$

5. [22] Let y_1 be the function given by $y_1(t) = t^2$. Given that y_1 is a solution of the differential equation

$$t^2 y''(t) - 4t\, y'(t) + 6\, y(t) = 0, \quad t > 0,$$

use reduction of order to find the general solution of the differential equation for $t > 0$.

Assume $y_2(t) = u(t)y_1(t) = t^2 u(t)$ is a second solution of the DE where $u = u(t)$ is a nonconstant function. Then

$$y_2'(t) = 2t\,u(t) + t^2 u'(t)$$

and

$$y_2''(t) = 2u(t) + 2tu'(t) + 2tu'(t) + t^2 u''(t)$$

$$= 2u(t) + 4tu'(t) + t^2 u''(t).$$

Substituting these expressions into the DE we want

$$t^2 y_2''(t) - 4t y_2'(t) + 6y_2(t) = 0$$

or

$$t^2\left(2u(t) + 4tu'(t) + t^2 u''(t)\right) - 4t\left(2t\,u(t) + t^2 u'(t)\right) + 6t^2 u(t) = 0.$$

Rearranging terms,

$$t^4 u''(t) + (4t^3 - 4t^3)u'(t) + (2t^2 - 8t^2 + 6t^2)u(t) = 0$$

or

$$t^4 u''(t) = 0.$$

Since $t > 0$, it follows that

$$u''(t) = 0$$

and integrating twice yields $u(t) = c_1 t + c_2$. Thus

$$y_2(t) = (c_1 t + c_2)t^2 = c_1 t^3 + c_2 t^2.$$

Taking $c_1 = 0$, $c_2 = 1$, gives $y_1(t) = t^2$.

Taking $c_1 = 1$, $c_2 = 0$, gives $y_2(t) = t^3$.

$$W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = 3t^4 - 2t^4 = t^4 > 0 \quad \text{on the interval } (0, \infty).$$

Therefore $y_1(t) = t^2$ and $y_2(t) = t^3$ form a fundamental set of solutions for the DE on the interval $(0, \infty)$. Hence the general solution is

$$\boxed{y(t) = c_1 t^2 + c_2 t^3} \qquad (c_1, c_2 \text{ arbitrary constants}).$$

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Section J

Exam I

Fall 2016

number of exams: 27

median score: 86

mean score: 77.9

standard deviation: 19.9

Distribution of Scores:

90 – 100	A	8
80 – 89	B	8
70 – 79	C	3
60 – 69	D	4
0 – 59	F	4