Mathematics 3304

Fall 2014

Exam III

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. Express all solutions in real-valued, simplified form.
8. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
9. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

	1	2	3	4	5	**Sum**
points earned						
maximum points	20	20	20	20	20	100

1.[20] Solve the initial value problem $y^{(4)} - y = 2\delta(t-1)$, $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

We use the Laplace transform method because of the Dirac impulse.

$$\mathcal{L}\{y^{(4)} - y\}(s) = \mathcal{L}\{2\delta(t-1)\}(s)$$

Applying linearity of Laplace transforms and formulas 8 and 11 in the Laplace transform table yields

$$s^4\mathcal{L}\{y\}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \mathcal{L}\{y\}(s) = 2e^{-s}.$$

Applying the initial conditions and simplifying yields

$$(s^4-1)\mathcal{L}\{y\}(s) = 2e^{-s}$$

So

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s^4-1}\right\}.$$

Using formula 10 of the Laplace transform table gives

$$y(t) = u_1(t)f(t-1) \qquad \text{where } f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^4-1}\right\}.$$

But $\dfrac{2}{s^4-1} = \dfrac{2}{(s^2-1)(s^2+1)} = \dfrac{As+B}{s^2-1} + \dfrac{Cs+D}{s^2+1}$ and a routine partial fraction decomposition calculation gives $A=0$, $B=1$, $C=0$, and $D=-1$. Therefore

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1} - \frac{1}{s^2+1}\right\} = \sinh(t) - \sin(t)$$

by formulas 6 and 3 in the Laplace transform table. Consequently

$$\boxed{y(t) = u_1(t)\left(\sinh(t-1) - \sin(t-1)\right)}.$$

2. (a) [16] Find the solution of the initial value problem $y'' + y = g(t)$, $y(0) = 1, y'(0) = -1$, where

$$g(t) = \begin{cases} 1 - t & \text{if } t < 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

(b) [4] Draw the graph of the solution on the interval $0 \leq t \leq 3\pi$.

(a) Observe that $g(t) = (1-t)(1-u_1(t)) = 1 - t + (t-1)u_1(t)$. Taking Laplace transforms of both sides of the DE

$$y'' + y = 1 - t + (t-1)u_1(t)$$

yields

$$s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) + \mathcal{L}\{y\}(s) = \frac{1}{s} - \frac{1}{s^2} + e^{-s}\cdot\frac{1}{s^2}$$

via formulas 8, 2, and 10 in the Laplace transform table. Applying the initial conditions and solving for $\mathcal{L}\{y\}(s)$ gives

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s^2(s^2+1)} + e^{-s}\frac{1}{s^2(s^2+1)}.$$

Routine partial fraction decomposition calculations show

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

and

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Therefore, linearity of the inverse Laplace transform and formulas 2, 3, and 10 yield

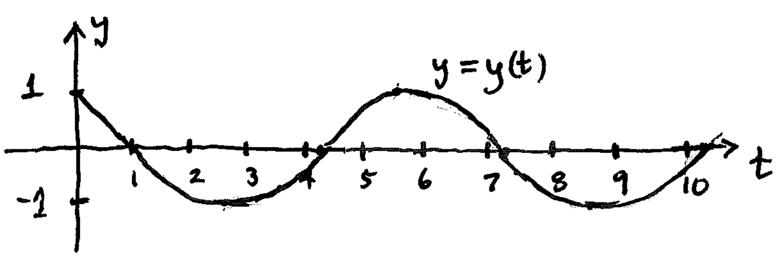
$$y(t) = \mathcal{L}^{-1}\left\{\cancel{\frac{s}{s^2+1}}\right\} - \mathcal{L}^{-1}\left\{\cancel{\frac{1}{s^2+1}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s} - \cancel{\frac{s}{s^2+1}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \cancel{\frac{1}{s^2+1}}\right\} + u_1(t)f(t-1)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin(t)$.

Consequently

$$\boxed{y(t) = 1 - t + u_1(t)\left(t - 1 - \sin(t-1)\right) = \begin{cases} 1-t & \text{if } t < 1, \\ -\sin(t-1) & \text{if } t \geq 1. \end{cases}}$$

(b)



3. [20] Solve the integro-differential equation

$$y' + 2y - \int_0^t 2y(\xi)\sin(t-\xi)\,d\xi = -\sin(t),$$

subject to the initial condition $y(0)=1$.

Since $\int_0^t f(\xi)g(t-\xi)d\xi$ is the convolution product $(f*g)(t)$, we recognize $\int_0^t y(\xi)\sin(t-\xi)d\xi$ as the convolution product of $f(t)=y(t)$ and $g(t)=\sin(t)$.

Consequently, the I-DE can be written as

$$y' + 2y - 2(y*\sin)(t) = -\sin(t).$$

Taking the Laplace transform of both sides and using linearity and formulas 8, 7, and 3 in the Laplace transform table gives

$$s\mathcal{L}\{y\}(s) - y(0) + 2\mathcal{L}\{y\}(s) - 2\mathcal{L}\{y*\sin\}(s) = -\mathcal{L}\{\sin(t)\}(s)$$

$$s\mathcal{L}\{y\}(s) - 1 + 2\mathcal{L}\{y\}(s) - 2\mathcal{L}\{y\}(s)\cdot\frac{1}{s^2+1} = \frac{-1}{s^2+1}.$$

Rearranging yields

$$\left(s+2-\frac{2}{s^2+1}\right)\mathcal{L}\{y\}(s) = 1-\frac{1}{s^2+1}$$

$$\left((s+2)(s^2+1)-2\right)\mathcal{L}\{y\}(s) = s^2+1-1$$

$$\mathcal{L}\{y\}(s) = \frac{s^2}{s^3+2s^2+s} = \frac{s}{s^2+2s+1} = \frac{s}{(s+1)^2}.$$

An easy partial fraction decomposition shows $\frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$.

Therefore formulas 1 and 9 in the table lead to

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} = \boxed{e^{-t} - te^{-t}}.$$

4. (a) [16] Transform the fourth order differential equation

$$(*) \qquad y^{(4)} - y'' - y = \sin\left(t^2\right)$$

into an equivalent system of first order differential equations.
(b) [4] Express this first order system in vector-matrix notation.

CAUTION: DO NOT ATTEMPT TO SOLVE EITHER THE SYSTEM OR THE ORIGINAL DIFFERENTIAL EQUATION!

(a) Suppose $y = y(t)$ solves equation $(*)$ and let

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \quad (= x_1'(t)) \\ x_3(t) &= y''(t) \quad (= x_2'(t)) \\ x_4(t) &= y'''(t) \quad (= x_3'(t)). \end{aligned}$$

Then from $(*)$ we obtain

$$x_4'(t) = y^{(4)}(t) = y''(t) + y(t) + \sin(t^2) = x_3(t) + x_1(t) + \sin(t^2).$$

Consequently $(*)$ is equivalent to the following system of 4 first-order DEs:

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ x_3'(t) = x_4(t) \\ x_4'(t) = x_1(t) + x_3(t) + \sin(t^2). \end{cases}$$

(b) The vector-matrix formulation of the first-order system above is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(t^2) \end{bmatrix}.$$

5. [20] Find the solution of the system of differential equations

$$x' = x + y$$
$$y' = -x + y$$

which satisfies the initial conditions $x(0) = 2$, $y(0) = -3$.

Let $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then the system of differential equations above is equivalent to $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Now $\vec{x}(t) = \vec{k}e^{rt}$ in $\vec{x}' = A\vec{x}$ leads to $r\vec{k} = A\vec{k}$.

$$0 = \det(A - rI) = \det\begin{bmatrix} 1-r & 1 \\ -1 & 1-r \end{bmatrix} = (1-r)^2 + 1 \quad \text{so} \quad (r-1)^2 = -1 \quad \text{and} \quad r = 1 \pm i.$$

An eigenvector $\vec{k}^{(1)}$ of A corresponding to the eigenvalue $r_1 = 1+i$ satisfies $(A - r_1 I)\vec{k} = \vec{0}$, i.e. $\begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or equivalently $\begin{cases} -ik_1 + k_2 = 0 \\ -k_1 - ik_2 = 0. \end{cases}$ Note that i times the second equation is equal to the first equation, so $k_2 = ik_1$ (where k_1 is arbitrary) is the solution of the system. Consequently $\vec{k}^{(1)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ ik_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ i \end{bmatrix}$.

We take $k_1 = 1$ for convenience. Therefore

$\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{r_1 t} = \begin{bmatrix} 1 \\ i \end{bmatrix}e^{(1+i)t}$ solves $\vec{x}' = A\vec{x}$. A fundamental set of solutions to $\vec{x}' = A\vec{x}$ is

$$\begin{cases} \tilde{\vec{x}}^{(1)}(t) = \operatorname{Re}\left(\vec{x}^{(1)}(t)\right) = \operatorname{Re}\left(e^t\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)(\cos(t) + i\sin(t))\right) = e^t\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} \\ \tilde{\vec{x}}^{(2)}(t) = \operatorname{Im}\left(\vec{x}^{(1)}(t)\right) = \operatorname{Im}\left(e^t\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)(\cos(t) + i\sin(t))\right) = e^t\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}. \end{cases}$$

Check: $W\left(\tilde{\vec{x}}^{(1)}(t), \tilde{\vec{x}}^{(2)}(t)\right) = \det\left[\tilde{\vec{x}}^{(1)}(t)\ \tilde{\vec{x}}^{(2)}(t)\right] = \det\begin{bmatrix} e^t\cos(t) & e^t\sin(t) \\ -e^t\sin(t) & e^t\cos(t) \end{bmatrix} = e^{2t} \neq 0.$

Therefore $\vec{x}(t) = c_1\tilde{\vec{x}}^{(1)}(t) + c_2\tilde{\vec{x}}^{(2)}(t)$ is the general solution of $\vec{x}' = A\vec{x}$. We want to choose c_1 and c_2 so the initial conditions are met:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{so} \quad c_1 = 2 \text{ and } c_2 = -3.$$

Therefore

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{x}(t) = 2\tilde{\vec{x}}^{(1)}(t) - 3\tilde{\vec{x}}^{(2)}(t) = \begin{bmatrix} e^t(2\cos(t) - 3\sin(t)) \\ e^t(-3\cos(t) - 2\sin(t)) \end{bmatrix}.$$

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. e^{at}	$\frac{1}{s-a}$
2. t^n	$\frac{n!}{s^{n+1}}$
3. $\sin(bt)$	$\frac{b}{s^2+b^2}$
4. $\cos(bt)$	$\frac{s}{s^2+b^2}$
5. $\cosh(at)$	$\frac{s}{s^2-a^2}$
6. $\sinh(at)$	$\frac{a}{s^2-a^2}$
7. $(f*g)(t)$	$F(s)G(s)$
8. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
9. $e^{ct} f(t)$	$F(s-c)$
10. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
11. $\delta(t-c)$	e^{-cs}

2014 Fall Semester, Math 3304 Hour Exam III
Instructor Grow, Section J

100		59		19	I
99		58	II	18	
98		57		17	
97		56		16	II
96		55	I	15	
95		54	I	14	
94		53		13	
93		52	I	12	
92		51	I	11	I
91		50		10	
90		49	I	9	
89		48		8	
88		47		7	
87		46		6	I
86		45	II	5	
85		44	I	4	
84		43	II	3	
83		42	I	2	
82		41		1	
81		40	I	0	
80	I	39			
79		38	I		
78		37			
77	I	36	I		
76		35	II		
75		34			
74		33			
73		32			
72		31			
71		30			
70		29	I		
69		28			
68	I	27			
67		26	I		
66		25			
65	I	24	I		
64		23			
63		22	II		
62	II	21			
61		20			
60					

Number taking exam: 34
Median: 43.0
Mean: 42.0
Standard Deviation: 18.5

Number receiving A's: 0
Number receiving B's: 1
Number receiving C's: 1
Number receiving D's: 4
Number receiving F's: 28