Chapter 1: Introduction

Sec. 1.1 Some Basic Mathematical Models; Direction Fields
HW p. 7: #1, 11, 22, 31  Due: Friday, August 27

Definition: An (ordinary) differential equation relates the derivatives of an unknown function of a single real variable.

Examples of ODEs: (Ask class for some examples)
\[ F = ma \quad \text{e.g.} \quad -mg = m \frac{dy}{dt} \quad \text{(models projectile motion if air resistance is neglected)} \]
\[ y' = 3x^2 \]
\[ y \frac{dy}{dx} + 4xy = x^2 \]

Examples of equations that are not ODEs:
\[ (x^3 + 5)' = 3x^2 \]
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \] (3-2 Wave Equation... PDE)
\[ x^3 + y^3 = 6xy \]

Differential equations are useful in modeling physical phenomena.

Exercise 1 (#24, p. 8) A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm³ of the drug enters the patient's bloodstream at a rate of 100 cm³/hr. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.1/hr.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug
that is present in the bloodstream at any time.

(b) How much of the drug is present in the bloodstream after a long time.

Solution: (a) Let \( A = A(t) \) be the amount (in mg) of the drug in the patient's bloodstream at time \( t \) (in hours).

\[
\frac{dA}{dt} = \text{Net rate of change of A} = \text{Net rate of inflow of A} - \text{Net rate of outflow of A}
\]

\[
\frac{dA}{dt} = \frac{5 \text{ mg}}{\text{cm}^2} \left( \frac{100 \text{ cm}^2}{\text{hr}} \right) - \frac{0.4 A}{\text{hr}} \left( \text{mg} \right)
\]

\[
\frac{dA}{dt} = 500 - 0.4A \quad \text{(units of both sides are mg/hr.)}
\]

(b) After a long time, the inflow and outflow rates will reach equilibrium, i.e., \( \frac{dA}{dt} \to 0 \) as \( t \to \infty \). The equilibrium amount of the drug in the patient's bloodstream will satisfy

\[
0 = 500 - 0.4A
\]

\[
\Rightarrow A = \frac{500}{0.4} = 1250 \text{ mg}
\]

We will learn how to solve first-order ODEs in Secs. 1.2 and 2.1-2.2. It is useful to study geometrically the behavior of first order ODEs \( y' = f(t, y) \) using its slope-field (or direction field).

Ex 2: (§3, p. 10) Draw a slope-field for \( f(t, y) = 1 - ty \).

(b) Based on the slope-field determine the behavior of the solution \( y = y(t) \) as \( t \to \infty \).
If the behavior depends on the initial value of \( y \) at \( t = 0 \), describe this dependency.

Solution: (a) A solution to (†) passing through the point \((3,1)\) in the \(ty\)-plane would have slope

\[
y' = f(3,1) = 1 - (2)(1) = -1
\]

there.

If we plot the slopes for solutions to (†) at a grid of points in the \(ty\)-plane, we obtain a slope field (or direction field) of (†).

(show Maple plot of slope field at this point.)

(b) From the slope field, it appears that \( y(t) \to 0 \) as \( t \to \infty \)

(This is independent of the initial value of \( y \) at \( t = 0 \).)
3. \( \frac{dy}{dx} = 1 - xy \)

(a) \( y(0) = 0 \)  
(b) \( y(-1) = 0 \)  
(c) \( y(2) = 2 \)  
(d) \( y(0) = -4 \)

\textbf{Figure 2.12}  Direction field for Problem 3