Chap. 2 First Order DEs

Sec. 2.1 Linear DEs; Method of Integrating Factors

HW p. 39: #7, 15, 28, 34   Due: Wed., Sept. 1
Schaums p.

1^{st} order DEs have the form: $y' = f(t,y)$

Linear 1^{st} order DEs have form: $a_1(t)y' + a_0(t)y = g(t)$

Dividing through by $a_1(t)$ we can place the 1^{st} order linear DE in "standard" form:

$$y' + p(t)y = q(t)$$

Ex 1 (Equivalent to #4, p. 39) Find the general solution on the interval $0<t<\infty$ of

$$(*) \qquad ty' + y = 3t\cos(2t).$$

Solution: Note that the product rule for derivatives implies that

$$\frac{d}{dt}(ty) = ty' + 1\cdot y = ty' + y.$$

Therefore the left member of $(*)$ is "exact"; i.e. the left member of $(*)$ is the derivative of the single expression ty. Hence $(*)$ can be rewritten as

$$\frac{d}{dt}(ty) = 3t\cos(2t).$$

Integrating both sides of this equation with respect to t yields

$$ty = \int \overbrace{3t}^{u}\,\overbrace{\cos(2t)\,dt}^{dV}$$

$$= 3t\left(\frac{\sin(2t)}{2}\right) - \int \frac{\sin(2t)}{2}3\,dt$$

$$ty = \frac{3}{2}t\sin(2t) + \frac{3}{4}\cos(2t) + c.$$

$$\therefore \boxed{y(t) = \frac{3}{2}\sin(2t) + \frac{3}{4t}\cos(2t) + \frac{c}{t}} \quad \text{on } 0<t<\infty.$$

Ex 2 (Equivalent to #10, p. 39) Find the general solution of

$$(*) \qquad y' - \frac{1}{t}y = t e^{-t}$$

on the interval $0 < t < \infty$.

Solution: Unlike the previous example, the left member of $(*)$ is not "exact". Following in the footsteps of Leonhard Euler (pronounced "oiler"), who wrote the first successful textbook on differential equations, we multiply both sides of $(*)$ by the "integrating factor" t^{-1} to make the left member exact:

$$t^{-1}\left(y' - \frac{1}{t}y\right) = t^{-1} \cdot t\, e^{-t}$$

t^{-1} is an "integrating factor" for $(*)$; it makes the left side "exact".

$$\left.\begin{aligned} t^{-1}y' - t^{-2}y &= e^{-t} \\ \frac{d}{dt}\left(t^{-1}y\right) &= e^{-t} \end{aligned}\right\}$$

Check: $\frac{d}{dt}(t^{-1}y) = t^{-1}y' - t^{-2}y$ ✓

Now we integrate both sides with respect to t:

$$t^{-1}y = \int e^{-t}\,dt = -e^{-t} + c$$

or

$$\boxed{y(t) = -te^{-t} + ct}$$

is the general solution of $(*)$ on $0 < t < \infty$.

Q: Can we always find an integrating factor for the 1st-order linear DE

$$y' + p(t)y = q(t)\ ?$$

A: (Euler) Yes, $\mu(t) = e^{\int p(t)\,dt}$ is an integrating factor. (See p. 36)

Check Ex 2 $y' - \frac{1}{t}y = te^{-t}$

An integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln(t)} = e^{\ln(t^{-1})} = t^{-1}$.

Here is an algorithm for solving first order linear DEs: $a_1(t)y' + a_0(t)y = g(t)$

1. Place the DE in standard form: $y' + p(t)y = q(t)$.
2. Compute an integrating factor $\mu(t) = e^{\int p(t)dt}$.
3. Multiply the DE in step 1 by the integrating factor $\mu(t)$.
4. Solve the resulting exact DE: $\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)q(t)$.

Note that the left member in step 4 should be the derivative of the product of the integrating factor $\mu(t)$ and the solution y we seek. You should always check this when you're solving such problems. It will help you identify errors you might have made in steps 1–3.

Ex 3 (Similar to #24, p. 40) Solve the initial value problem.

$$ty' + (t+1)y = 2te^{-t}, \quad y(1) = 0.$$

Solution: Note that the DE is first order linear: $a_1(t)y' + a_0(t)y = g(t)$ where $a_1(t) = t$, $a_0(t) = t+1$, and $g(t) = 2te^{-t}$.

Step 1: $y' + \left(\frac{t+1}{t}\right)y = 2e^{-t}$

Step 2: $\mu(t) = e^{\int p(t)dt} = e^{\int (1+\frac{1}{t})dt} = e^{t+\ln(t)} = e^{t} \cdot e^{\ln(t)} = te^{t}$.

Step 3: $te^{t}\left[y' + \left(\frac{t+1}{t}\right)y\right] = te^{t}(2e^{-t})$

$$te^{t}y' + (t+1)e^{t}y = 2t$$

$$\frac{d}{dt}\left[te^{t} y \right] = 2t$$

Check:

$$\frac{d}{dt}[te^{t}y] = te^{t}y' + (te^{t})'y$$
$$= te^{t}y' + (te^{t} + 1\cdot e^{t})y$$
$$= te^{t}y' + (t+1)e^{t}y \quad ✓$$

Step 4:

$$te^{t}y = \int 2t\,dt$$

arbitrary constant

$$te^{t}y = t^2 + c$$

$\therefore$ $y(t) = te^{-t} + ct^{-1}e^{-t}$ is the general solution of the DE on $0 < t < \infty$. We want to choose c so $y(1) = 0$.

$$0 = y(1) = e^{-1} + ce^{-1} \Rightarrow c = -1.$$

$\boxed{y(t) = te^{-t} - t^{-1}e^{-t}}$ solves the I.V.P.