2. First Order DEs

Sec. 2.1 Linear DEs; Method of Integrating Factors

HW p. 39: #7, 15, 28, 34 Due: Wed., Sept. 1
Schaums p.

1st order DEs have the form:  \( y' = f(t, y) \)

Linear 1st order DEs have form:  \( q(t)y' + a(t)y = g(t) \)

Dividing through by \( q(t) \) we can place the 1st order linear DE in "standard" form

\[ y' + p(t)y = q(t) \]

Ex 1| (Equivalent to #4, p. 39) Find the general solution on the interval \(-\infty < t < \infty\) of

(*)  \[ ty' + y = 3t \cos(2t) \]

Solution: Note that the product rule for derivatives implies that

\[ \frac{d}{dt}(ty) = ty' + 1y = ty' + y \]

Therefore the left member of (*) is "exact"; i.e. the left member of (*) is the derivative of the single expression \( ty \). Hence (*) can be rewritten as

\[ \frac{d}{dt}(ty) = 3t \cos(2t) \]

Integrating both sides of this equation with respect to \( t \) yields

\[ ty = \int \frac{3t \cos(2t)}{3t \cos(2t)} \, dt \]

\[ = 3t \left( \frac{\sin(2t)}{2} \right) - \int \frac{\sin(2t) \, 3 \, dt}{2} \]

\[ ty = \frac{3}{2} t \sin(2t) + \frac{3}{4} \cos(2t) + c \]

\[ \therefore y(t) = \frac{3}{2} \sin(2t) + \frac{3}{4} \cos(2t) + \frac{c}{t} \quad \text{on} \quad -\infty < t < \infty \]
Ex. 2] (§ 10, p. 39) Find the general solution of

\[ y' - \frac{1}{t} y = t e^{-t} \]

on the interval \(0 < t < \infty\).

Solution: Unlike the previous example, the left member of (\(*)\) is not "exact".

Following in the footsteps of Leonard Euler (pronounced "oi ler"), who wrote the first textbook on differential equations, we multiply both sides of (\*) by the "integrating factor" \(t^{-1}\) to make the left member exact:

\[ t^{-2} (y' - \frac{1}{t} y) = t^{-2} t e^{-t} \]

\[ t^{-2} y' - t^{-1} y = e^{-t} \]

Check:

\[ \frac{d}{dt} (t^{-1} y) = t^{-1} y' - t^{-2} y \]

Now we integrate both sides with respect to \(t\):

\[ t^{-1} y = \int e^{-t} dt = -e^{-t} + c \]

or

\[ y(t) = -te^{-t} + ct \]

is the general solution of (\*) on \(0 < t < \infty\).

Q: Can we always find an integrating factor for the 1st-order linear DE

\[ y' + p(t)y = q(t) \]

A: (Euler) Yes, \( \mu(t) = e^{\int p(t) dt} \) is an integrating factor. (§ 10, p. 36)
CheckEx: \[ y' - \frac{1}{t}y = te^t \]

An integrating factor is \( \mu(t) = e^{\int \frac{1}{t} dt} = e^{\int \frac{1}{t} dt - \ln(t)} = e^{\ln(t^*)} = t^* \).

Here is an algorithm for solving first order linear DEs: \( q(t)y' + a(t)y = g(t) \)

1. Place the DE in standard form: \( y' + pt(t)y = q(t) \).
2. Compute an integrating factor \( \mu(t) = e^{\int pt(t) dt} \).
3. Multiply the DE in step 1 by the integrating factor \( \mu(t) \).
4. Solve the resulting exact DE: \[ \frac{d}{dt} [\mu(t)y] = \mu(t)q(t) \]

Note that the left member in step 4 should be the derivative of the product of the integrating factor \( \mu(t) \) and the solution \( y \) we seek. You should always check this when you're solving such problems. It will help you identify errors you might have made in steps 1–3.

Ex 3 \( \) (Similar to #24, p. 20) Solve the initial value problem:

\[ ty' + (t+1)y = 2te^{-t}, \quad y(1) = 0. \]

Solution: Note that the DE is first order linear: \( a(t)y' + a(t)y = g(t) \), where \( a(t) = t, \quad a(1) = 1 + t, \quad a(2) = 2e^{-t} \).

Step 1: \[ y' + \left(\frac{t+1}{t}\right)y = 2e^{-t} \]

Step 2: \[ \mu(t) = e^{\int \frac{t+1}{t} dt} = e^{\int\left(1 + \frac{1}{t}\right) dt} = e^{t + \ln(t)} = te^t \]

Step 3: \[ te^t \left[ y' + \left(\frac{t+1}{t}\right)y \right] = te^t (2e^{-t}) \]

\[ te^t y' + (t+1)e^t y = 2t \]
\[ \frac{d}{dt} \left[ te^y \right] = 2t \]

Step 1:
\[ te^y = \int 2t \, dt \]
\[ te^y = t^2 + C \]
\text{arbitrary constant}

Check:
\[ \frac{d}{dt} [te^y] = te^y + (te^t)y \]
\[ = t^2e^y + (t+1)e^t y \]
\[ = t^2e^y + (t+1)e^y \checkmark \]

\[ y(t) = te^t + Cte^{-t} \]

is the general solution of the DE on \( 0 < t < \infty \). We want to choose \( C \) so \( y(1) = 0 \).

\[ C = y(1) = e^1 + C e^{-1} \implies C = -1. \]

\[ y(t) = te^t - te^{-t} \]

solves the IVP.