Sec. 2.2  Separable Equations

HW p.47: # 1, 11, 21, 30  Due: Friday, Sept 3

Schaum's: pp. 21-29

1st order DEs have the form  \( y' = f(t, y) \)
1st order linear DEs have form  \( y' = -p(t)y + q(t) \)  \( \text{(or} \quad \frac{d}{dt}(ay + q(t)y) = f(t)) \)
1st order separable DEs have form  \( y' = g(t)h(y) \)

To solve separable DEs we rewrite the eqn. as
\[
\frac{dy}{h(y)} = g(t)dt
\]
and separate variables
\[
\int \frac{dy}{h(y)} = \int g(t)dt
\]

Then integrate
\[
\int \frac{dy}{h(y)} = \int g(t)dt
\]

Ex1 (cf. #23, p.48) Find the general solution of

\[ y' = 2y^2 + xy^2. \]

Soln:  \( y' = y^2(2+x) \)  separable.

\[
\frac{dy}{y^2} = (2+x)dx
\]

\[
\int \frac{dy}{y^2} = \int (2+x)dx
\]

\[
-\frac{1}{y} = 2x + \frac{x^2}{2} + C
\]

This equation defines \( y \) implicitly as a function of \( x \).

\( \text{(Implicit Solution)} \)

\[
\frac{-\frac{1}{y}}{\frac{x^2}{2} + 2x + C} = y
\]

(Explicit solution)
Ex. 21 (#22, p. 48) Solve the IVP

$$y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0$$

and determine the interval in which the solution is valid.

**Solu:** \( \frac{dy}{dx} = 3x^2 \left( \frac{1}{3y^2 - 4} \right) \) (Variables are separable)

\[
\int (3y^2 - 4)\,dy = \int 3x^2\,dx
\]

\[
y^3 - 4y = x^3 + C \quad \leftarrow \text{This defines } y \text{ implicitly as a function of } x. \text{ It is extremely difficult (but not impossible!) to solve for } y \text{ as an explicit function of } x.
\]

We will proceed with the implicit form of the soln.

When \( x = 1 \), we want \( y = 0 \). Therefore

\[
0 - 4(0) = 1 + C \quad \Rightarrow \quad C = -1
\]

\[
y^3 - 4y = x^3 - 1
\]

In order to determine the largest interval \( a < x < b \) containing \( x = 1 \) in which the solution is valid, we examine those points where the derivative of \( y \) is undefined. Returning to the DE, we see that these occur when \( 3y^2 - 4 = 0 \) \( \Rightarrow \) \( y = \pm \sqrt{\frac{4}{3}} = \pm \frac{2}{\sqrt{3}} \).

If \( y = \frac{2}{\sqrt{3}} \) then \( \left( \frac{2}{\sqrt{3}} \right)^3 - 4\left( \frac{2}{\sqrt{3}} \right) = x^3 - 1 \) \( \Rightarrow \)

\[
\frac{8}{3\sqrt{3}} - 8 = x^3 \Rightarrow \frac{8 - 24 + 3\sqrt{3}}{3\sqrt{3}} = x
\]

\[
\Rightarrow \quad x = \sqrt[3]{\frac{3\sqrt{3} - 16}{3\sqrt{3}}} \approx -1.276337
\]

If \( y = -\frac{2}{\sqrt{3}} \) then a similar calculation gives \( x = \sqrt[3]{\frac{3\sqrt{3} + 16}{3\sqrt{3}}} \approx 1.597810 \)

The solution is valid on \( \boxed{-1.276337 < x < 1.597810} \) (approximately).
Ex 3] (cf. #30, p. 49) Find the general solution to the DE
\[
y' = \frac{y-x}{y+x},
\]

Sdn: This DE is not separable since \( \frac{y-x}{y+x} \neq g(x) h(y) \). Note, however, that the DE is equivalent to
\[
(1) \quad y' = \frac{\frac{y}{x} - 1}{\frac{y}{x} + 1}.
\]

The RHS depends on the ratio \( \frac{y}{x} \). This suggests we make the change of variable \( v = \frac{y}{x} \), or equivalently \( y = vx \). Then \( y' = v'x + v \) so substituting in (1) yields
\[
v'x + v = \frac{\frac{v}{x} - 1}{\frac{v}{x} + 1}.
\]

\[\Rightarrow\]
\[
v' = \left( \frac{\frac{v}{x} - 1}{\frac{v}{x} + 1} - v \right) \frac{1}{x} \]
\[\frac{1}{y(v)} \frac{1}{h(x)} \]

separable

\[
\frac{dv}{dx} = \left( \frac{\frac{v}{x} - 1 - \frac{v(v+1)}{v+1}}{v+1} \right) \frac{1}{x} \quad \Rightarrow \quad \frac{v+1}{v^2+1} \frac{dv}{dx} = -\frac{1}{x} dx
\]

\[
\int \frac{v \, dv}{v^2+1} + \int \frac{1 \, dx}{v^2+1} = -\int \frac{1}{x} \, dx \quad \Rightarrow \quad \frac{1}{2} \ln(v^2+1) + \arctan(v) = -\ln|x| + C
\]

But \( v = \frac{y}{x} \) so \( \frac{1}{2} \ln\left( \frac{y^2}{x^2}+1 \right) + \frac{1}{2} \ln(x^2) + \arctan\left( \frac{y}{x} \right) = C \)

or
\[
\ln\sqrt{x^2 + y^2} + \arctan\left( \frac{y}{x} \right) = C \quad \text{(Implicit Form)}
\]

In polar coordinates, this is \( \ln(r) + \theta = C \) or \( r = Ae^{-\theta} \) \( (A=e^C) \)

(implicit logarithmic spiral)