Sec. 2.3 Modeling with 1^{st} Order DEs

HW p.59: #4, 13, 16, 23

Schaum's pp. 50 – 68

When building DEs to model physical processes, the following principles are useful:

① $\frac{dy}{dt}$ represents the rate of change of $y = y(t)$ with respect to t.

② $\frac{dy}{dt}$ represents the slope of the line tangent to the graph of $y = y(t)$ at a general point (t, y).

<u>Ex 1</u> (#2, p.60) A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ grams per liter of salt enters the tank at a rate of 2 liters per minute, and the well-stirred mixture leaves the tank at the same rate.

(a) Find an expression in terms of γ for the amount of salt in the tank at any time t.

(b) Find the limiting amount of salt in the tank as $t \to \infty$.

<u>Soln</u>: Let $A(t)$ denote the amount of salt (in grams) at any time t (measured in minutes). We use the principle

Net rate of change of ~~salt~~ = Rate at which salt is entering − Rate at which salt is leaving

$$\frac{dA}{dt} = \left(\frac{\gamma\, g}{\cancel{\ell}}\right)\left(\frac{2\cancel{\ell}}{\text{min}}\right) - \left(\frac{A\, g}{120\,\cancel{\ell}}\right)\left(\frac{2\cancel{\ell}}{\text{min}}\right)$$

IVP that models the salt in tank at time t

$$\boxed{\begin{aligned} \frac{dA}{dt} &= 2\gamma - \frac{1}{60}A \\ A(0) &= 0 \end{aligned}}$$

(all terms have units of grams per minute)

$$\frac{dA}{dt} + \frac{1}{60}A = 2\gamma \qquad (\text{linear, } 1^{st} \text{ order})$$

$$\text{I.F.}: \quad \mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{60}dt} = e^{t/60}$$

$$\underbrace{e^{t/60}\frac{dA}{dt} + \frac{1}{60}e^{t/60}A}_{\text{Exact}} = 2\gamma e^{t/60}$$

$$\frac{d}{dt}\left(e^{t/60}A\right) = 2\gamma e^{t/60}$$

$$e^{t/60}A = \int 2\gamma e^{t/60}\,dt = 120\gamma e^{t/60} + C$$

$$A(t) = 120\gamma + Ce^{-t/60}$$

$$0 = A(0) = 120\gamma + C \implies C = -120\gamma$$

(a) $\therefore$ $\boxed{A(t) = 120\gamma - 120\gamma e^{-t/60}}$

Expression in terms of γ that gives amount of salt (in grams) in tank at time t.

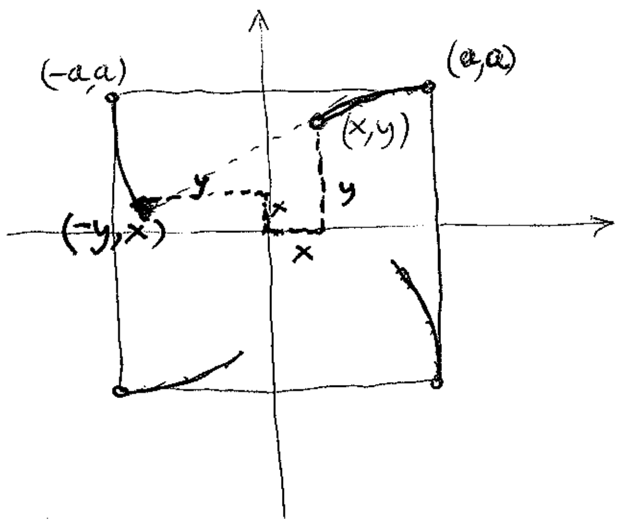
(b) $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(120\gamma - 120\gamma e^{-t/60}\right) = \boxed{120\gamma}$ grams

is the limiting amount of salt in the tank as $t \to \infty$.

B on handout on last page.

Ex 2 (The fly problem ... a pursuit problem) Four flies initially sit at the four corners of a square table, facing inward. At the same instant each fly begins walking at the same rate toward the fly on its right. Find the path that each fly follows.

Soln:



When the first fly (i.e. the one starting in QI) is at position (x, y) then by symmetry the second fly (i.e. the one starting in QII) is at position $(-y, x)$. In pursuit problems, the line-of-sight of the pursuer is tangent to the path of the pursuer. Therefore

$$\frac{dy}{dx} = \text{slope of the tangent to the first fly's path} = \frac{\Delta y}{\Delta x} = \frac{y-x}{x-(-y)}$$

or $$\boxed{\frac{dy}{dx} = \frac{y-x}{y+x}, \qquad y(a) = a}$$

is an IVP that models the first fly's path. In Sec. 2.2 we found the general solution of this DE:

$$\ln\sqrt{x^2+y^2} + \text{Arctan}\left(y/x\right) = c$$

or in polar coordinates

$$\ln(r) + \theta = c$$

$$\Updownarrow$$

$$r = Ae^{-\theta} \qquad (A = e^c)$$

When $r = \sqrt{2}a$ we have $\theta = \pi/4$.

$$\sqrt{2}a = Ae^{-\pi/4}$$

$$\Rightarrow A = \sqrt{2}ae^{\pi/4}$$

$$\therefore \boxed{r(\theta) = \sqrt{2}ae^{\frac{\pi}{4}-\theta}} \qquad \text{(Fly in QI's trajectory)}$$

Other trajectories:

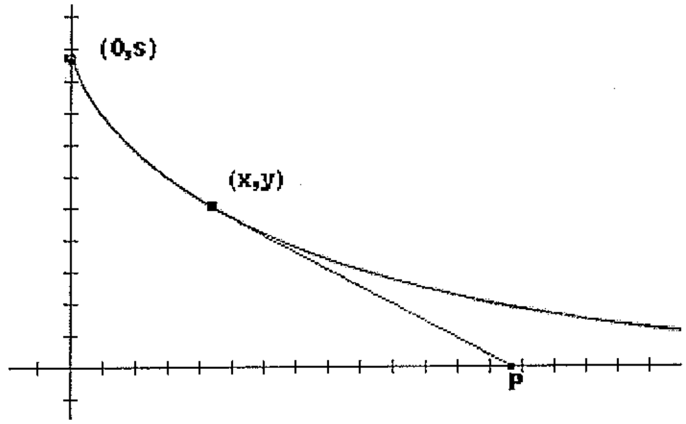
$$r(\theta) = \sqrt{2}ae^{\frac{3\pi}{4}-\theta} \qquad \text{(Fly in QII)}$$

$$r(\theta) = \sqrt{2}ae^{\frac{5\pi}{4}-\theta} \qquad \text{(Fly in QIII)}$$

$$r(\theta) = \sqrt{2}ae^{\frac{7\pi}{4}-\theta} \qquad \text{(Fly in QIV)}$$

Supplement to Section 2.3: Pursuit Problems

A. A person P, starting at the origin, moves in the direction of the positive $x-$ axis, pulling a weight along a curve C as shown in the figure below. The weight, initially located on the $y-$axis at $(0,s)$, is pulled by a rope of constant length s which is kept taut throughout the motion. Determine an equation for the path of motion C. (Hint: The rope is always tangent to C.)



B. Four flies initially sit at the four corners of a square table, facing inward. At the same instant, each fly begins walking at the same rate toward the fly on its right. Find the path that each fly follows. (Hint: The line-of-sight for a pursuing fly is always tangent to the path of the pursuer.)