Sec. 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients.

HW p.183: # 1, 4, 15, 29 Due: Mon., Feb. 25

Schaum's: pp. 94-102.

Q: What is the form of the most general solution to

$$(*) \qquad y'' + p(t)y' + q(t)y = g(t)\ ?$$

A: (Theorem 3.5.2, p.175)

$$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

where $\{y_1, y_2\}$ is a F.S.S. to the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$, c_1 and c_2 are arbitrary constants, and y_p is any particular solution (i.e. one which has no arbitrary constants) of the nonhomogeneous equation $(*)$.

In this section and the next we will study methods for finding a particular solution of $(*)$. We first study the method of undetermined coefficients which is a guess-and-check process. Table 3.5.1 on p.181 will be useful in determining judicious "guesses" for y_p based on $g(t)$.

$g(t)$	Trial form for y_p
$P_n(t) = a_n t^n + \dots + a_1 t + a_0$	$t^s(A_n t^n + \dots + A_1 t + A_0)$
$P_n(t)e^{\alpha t}$	$t^s(A_n t^n + \dots + A_1 t + A_0)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin(\beta t)$ or $P_n(t)e^{\alpha t}\cos(\beta t)$	$t^s[(A_n t^n + \dots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + (B_n t^n + \dots + B_1 t + B_0)e^{\alpha t}\sin(\beta t)]$

(Here s is the multiplicity as a root of the characteristic equation of 0, α, or $\alpha + i\beta$, respectively.)

Ex 1| Find the general solution of $y''-y'-2y=8e^{3t}$.

Step 1: Solve the associated homogeneous equation.

$y=e^{rt}$ in $y''-y'-2y=0$ leads to $r^2-r-2=0 \Rightarrow (r-2)(r+1)=0$
$\Rightarrow r=2$ or $r=-1$. Thus $y_h(t)=c_1e^{2t}+c_2e^{-t}$ is the gen. soln. of the assoc. hom. eqn.

Step 2: Find a particular solution of the nonhomogeneous equation.
(We use table 3.5.1 on p. 181.) $g(t)=8e^{3t}$ is the product of a polynomial of degree zero times an exponential function:

$$g(t)=P_0(t)e^{\alpha t}=a_0e^{\alpha t}=8e^{3t}.$$

Let s = multiplicity of $\alpha=3$ as a root of the characteristic equation $r^2-r-2=0$.
Then $s=0$. A trial form for a particular solution is

$$y_p(t)=t^s(A_nt^n+\dots+A_1t+A_0)e^{\alpha t}=t^0(A_0)e^{3t}=Ae^{3t}$$

where A is a constant to be determined. Then

$$y_p'=3Ae^{3t} \quad \text{and} \quad y_p''=9Ae^{3t}, \quad \text{so substituting}$$

$$y_p''-y_p'-2y_p \overset{\text{Want}}{=} 8e^{3t}$$

$$9Ae^{3t}-3Ae^{3t}-2Ae^{3t}=8e^{3t}$$

$$4A\cancel{e^{3t}}=8\cancel{e^{3t}} \quad \Rightarrow \quad A=2$$

$$\therefore\ y_p(t)=2e^{3t}.$$

Step 3: Write the general solution $y=y_h+y_p$ of the nonhomogeneous eqn.

$$\boxed{y(t)=c_1e^{2t}+c_2e^{-t}+2e^{3t}}.$$

Ex 2 Find the general solution of $y'' - 3y' = t^2$.

Step 1: Solve the associated homogeneous equation.

$y = e^{rt}$ in $y'' - 3y' = 0$ leads to $r^2 - 3r = 0 \Rightarrow r(r-3) = 0 \Rightarrow$ $r = 0$ or $r = 3$. Therefore $y_h(t) = c_1 + c_2 e^{3t}$ is the gen. soln. of the hom. eqn.

Step 2: Find a particular solution of the nonhomogeneous equation.

(We use table 3.5.1 on p. 181.) $g(t) = t^2$ is a polynomial of degree 2: $P_2(t) = 1t^2 + 0t + 0$. Let s = multiplicity of $\alpha = 0$ as a root of the characteristic equation $r^2 - 3r = 0$. Then $s = 1$ so a trial form for a particular solution is

$$y_p(t) = t^s(A_2t^2 + A_1t + A_0) = t(At^2 + Bt + C) = At^3 + Bt^2 + Ct$$

where A, B, C are constants to be determined. Then

$$y_p' = 3At^2 + 2Bt + C$$
$$y_p'' = 6At + 2B.$$

$$y_p'' - 3y_p' \overset{\text{Want}}{=} t^2$$

$$(6At + 2B) - 3(3At^2 + 2Bt + C) = t^2$$

$$-9At^2 + (6A - 6B)t + (2B - 3C) = 1 \cdot t^2 + 0 \cdot t + 0 \cdot 1$$

$\therefore \; -9A = 1, \quad 6A - 6B = 0, \;$ and $\; 2B - 3C = 0.$

$\Rightarrow \; A = -\frac{1}{9}, \quad B = A = -\frac{1}{9}, \quad C = \frac{2}{3}B = -\frac{2}{27}$

$\therefore \; y_p(t) = -\frac{1}{9}t^3 - \frac{1}{9}t^2 - \frac{2}{27}t$

Step 3: Write the general solution $y = y_h + y_p$ of the nonhom. equation.

$$\boxed{y(t) = c_1 + c_2e^{3t} - \frac{1}{9}t^3 - \frac{1}{9}t^2 - \frac{2}{27}t}.$$

Ex 3 Find the general solution of $y'' + 2y' + y = e^{-t}$.

Step 1: Solve the associated homogeneous equation.

$y = e^{rt}$ in $y'' + 2y' + y = 0$ leads to $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$
$\Rightarrow r = -1$ (multiplicity two). Therefore $y_h(t) = c_1 e^{-t} + c_2 t e^{-t}$ is the gen. soln. of the assoc. hom. eqn.

Step 2: Find a particular solution of the nonhomogeneous equation.
(We use table 3.5.1 on p. 181.) $g(t) = e^{-t}$ is the product of a polynomial of degree zero and an exponential function:

$$g(t) = P_0(t) e^{\alpha t} = a_0 e^{\alpha t} = 1 \cdot e^{-t}.$$

Let $s =$ multiplicity of $\alpha = -1$ as a root of the characteristic equation $r^2 + 2r + 1 = 0$. Then $s = 2$ so a trial form for a particular solution is

$$y_p(t) = t^s (A_n t^n + \dots + A_1 t + A_0) e^{\alpha t} = t^2 (A_0) e^{-t} = A t^2 e^{-t}$$

where A is a constant to be determined. Then

$$y_p' = 2Ate^{-t} - At^2 e^{-t} = (2t - t^2) A e^{-t}$$
$$y_p'' = (2 - 2t) A e^{-t} - (2t - t^2) A e^{-t} = (2 - 4t + t^2) A e^{-t}$$ so substituting

$$y_p'' + 2y_p' + y_p \overset{\text{Want}}{=} e^{-t}$$

$$(2 - 4t + t^2) A \cancel{e^{-t}} + 2(2t - t^2) A \cancel{e^{-t}} + A t^2 \cancel{e^{-t}} = \cancel{e^{-t}}$$

$$\underbrace{(A - 2A + A)}_{0} t^2 + (\underbrace{-4A + 4A}_{0}) t + 2A = 0t^2 + 0t + 1t^0$$

$$\Rightarrow A = 1/2$$

$$\therefore y_p(t) = \frac{1}{2} t^2 e^{-t}.$$

Step 3: Write the general solution $y = y_h + y_p$ of the nonhom. equation.

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}$$

Ex 4 Find the general solution of $\frac{1}{4}y'' + 16y = 8\cos(8t)$.

Step 1: Solve the associated homogeneous equation.

$y = e^{rt}$ in $\frac{1}{4}y'' + 16y = 0$ leads to $\frac{1}{4}r^2 + 16 = 0 \Rightarrow r^2 = -64$

$\Rightarrow r = \pm 8i$. Therefore $y_h(t) = c_1\cos(8t) + c_2\sin(8t)$ is the gen. soln. of the assoc. hom. eqn.

Step 2: Find a particular solution of the nonhomogeneous equation.

(We use table 3.5.1 on p. 181.) $g(t) = 8\cos(8t)$ is a product of a polynomial of degree zero, a constant exponential function, and a cosine function:

$$g(t) = P_0(t)e^{\alpha t}\cos(\beta t) = a_0 e^{0t}\cos(8t)$$

Let s = multiplicity of $\alpha + i\beta = 0 + i8$ as a root of the characteristic equation $\frac{1}{4}r^2 + 16 = 0$. Then $s = 1$ so a trial form for a particular solution is

$$y_p(t) = t^s\left[(A_n t^n + \dots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + (B_n t^n + \dots + B_1 t + B_0)e^{\alpha t}\sin(\beta t)\right]$$
$$= t\left[A\cos(8t) + B\sin(8t)\right]. \qquad (A, B \text{ constants to be determined})$$

Then
$$y_p' = 1\cdot\left[A\cos(8t) + B\sin(8t)\right] + t\left[-8A\sin(8t) + 8B\cos(8t)\right]$$
$$y_p'' = -8A\sin(8t) + 8B\cos(8t) + 1\left[-8A\sin(8t) + 8B\cos(8t)\right] + t\left[-64A\cos(8t) - 64B\sin(8t)\right]$$
$$= -16A\sin(8t) + 16B\cos(8t) - 64t\left[A\cos(8t) + B\sin(8t)\right]$$

Substituting: $\frac{1}{4}y_p'' + 16y_p \overset{\text{Want}}{=} 8\cos(8t)$

$$-4A\sin(8t) + 4B\cos(8t) \cancel{- 16t\left[A\cos(8t) + B\sin(8t)\right]} + \cancel{16t\left[A\cos(8t) + B\sin(8t)\right]} = 8\cos(8t) + 0\sin(8t)$$

$\Rightarrow -4A = 0$ and $4B = 8 \Rightarrow A = 0$ and $B = 2$.

$\therefore \quad y_p(t) = 2t\sin(8t)$.

Step 3: The gen. soln. $y = y_h + y_p$ of the nonhomogeneous equation is

$$\boxed{y(t) = c_1\cos(8t) + c_2\sin(8t) + 2t\sin(8t)}.$$