Sec. 4.1 General Theory of nth Order Linear Equations

HW p. 224: #3, 7, 15, 19 Due: Fri., Oct. 8
Schaum's: pp. 89-93

The theory associated with the general nth order linear equation

\[ \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = g(t) \]

is very similar to that for second order linear equations which we studied in Sec. 3.2.

Q: When is a linear nth order IVP consisting of \( \frac{d^n y}{dt^n} \) and the initial conditions

\[ y(t_0) = y_0, \ y'(t_0) = y_1, \ldots, \ y^{(n-2)}(t_0) = y_{n-2} \]

guaranteed to have exactly one solution?

A: Theorem 4.1.1, p. 220 (Existence and Uniqueness Theorem)

Don't write on board. Ask students to read along on p. 220.

\[ \begin{cases} 
\text{If the functions } p_1, p_2, \ldots, p_n, \text{ and } g \text{ are continuous on the open interval } I, \text{ then there exists exactly one solution } y = y(t) \text{ of the DE (2) that also satisfies the initial conditions (3). This solution exists throughout the interval } I. 
\end{cases} \]

Ex 1 (5, p. 224) Determine intervals in which solutions to

\[ (t-1)y^{(4)} + (t+1)y'' + tan(t)y = 0 \]

are sure to exist.

Solution: In order to make use of the existence/uniqueness theorem 4.1.1 we must place the DE in standard form:

\[ y^{(4)} + p_1(t)y^{(3)} + p_2(t)y'' + p_3(t)y' + p_4(t)y = g(t) \].
Our DE is equivalent to

\[ y^{(4)} + \frac{t+1}{t-1} y'' + \frac{\tan(t)}{t-1} y = 0. \]

\[ p_1(t) = 0 \text{ is continuous on } (-\infty, \infty), \]

\[ p_2(t) = \frac{t+1}{t-1} \text{ is continuous on } (-\infty, 1) \text{ and } (1, \infty), \]

\[ p_3(t) = 0 \text{ is continuous on } (-\infty, \infty), \]

\[ p_4(t) = \frac{\tan(t)}{t-1} \text{ is continuous on } (-\frac{\pi}{2}, 1) \text{ and } (1, \frac{\pi}{2}) \text{ and } (\frac{\pi}{2}, \frac{3\pi}{2}) \text{ etc.}, \]

\[ q(t) = 0 \text{ is continuous on } (-\infty, \infty). \]

The DE is guaranteed to have solutions on any of the following intervals:

\[ \ldots, (-\frac{5\pi}{2}, -\frac{3\pi}{2}), (-\frac{3\pi}{2}, -\frac{\pi}{2}), (-\frac{\pi}{2}, 1), (1, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{2}), \ldots \]

Q2: What is the form of the most general solution of the linear, homogeneous \( n \)th order DE

\[ (2) \quad \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \ldots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = q(t) ? \]

Definition: If \( f_1, f_2, \ldots, f_n \) are \((n-1)\)-times differentiable functions then their Wronskian is the function

\[ W(f_1, f_2, \ldots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \ldots & f_n(t) \\ f_1'(t) & f_2'(t) & \ldots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \ldots & f_n^{(n-1)}(t) \end{vmatrix}. \]
Ex. 2] (§ 14, p. 224) Verify that the functions
\[ f_1(t) = 1, \quad f_2(t) = t, \quad f_3(t) = e^{-t}, \quad f_4(t) = te^{-t} \]
are solutions of the DE
\[ y^{(4)} + 2y^{(2)} + y = 0, \]
and determine their Wronskian.

Solution: Clearly \( f_1^{(4)} = f_1^{(3)} = f_1^{(2)} = 0 \) so \( f_1^{(4)} + 2f_1^{(2)} + f_1 = 0 \).
A similar argument shows \( f_2 \) satisfies \( f_2^{(4)} + 2f_2^{(2)} + f_2 = 0 \). Also
\[
f_3^{(4)} + 2f_3^{(2)} + f_3 = e^{-t} + 2(-e^{-t}) + e^{-t} = 0.
\]
\[
f_4^{(4)} + 2f_4^{(2)} + f_4 = (t-4)e^{-t} + 2(3-t)e^{-t} + (t-2)e^{-t}
\]
\[= (t-4+6-2t+t-2)e^{-t}\]
\[= 0\]

\[
W(f_1, f_2, f_3, f_4)(t) = \begin{vmatrix}
1 & t & e^{-t} & te^{-t} \\
0 & 1 & -e^{-t} & (t-1)e^{-t} \\
0 & 0 & e^{-t} & (t-2)e^{-t} \\
0 & 0 & 0 & (3-t)e^{-t}
\end{vmatrix}
\]
(Add row 3 to row 4.)

\[
\begin{vmatrix}
1 & t & e^{-t} & te^{-t} \\
0 & 1 & -e^{-t} & (t-1)e^{-t} \\
0 & 0 & e^{-t} & (t-2)e^{-t} \\
0 & 0 & 0 & e^{-t}
\end{vmatrix}
\]
(The determinant of an upper triangular matrix is the product of the elements on the principal diagonal.)

\[= (1)(1)(e^{-t})e^{-t} = \frac{-2t}{e}\]
A2: The most general solution of

\[ \frac{d^n}{dt^n} + p(t)\frac{d^{n-1}}{dt^{n-1}} + \ldots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t) \]

on an interval \( I \) is of the form

\[ y = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t) + \Psi(t) \]

where \( y_1, \ldots, y_n \) is a set of \( n \) solutions to the associated homogeneous equation of order \( n \),

\[ y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = 0, \]

such that \( W(y_1, y_2, \ldots, y_n)(t) \neq 0 \) on \( I \);

\( c_1, c_2, \ldots, c_n \) are arbitrary constants;

\( \Psi(t) \) is a particular solution of (2) on \( I \).

(See Theorem 4.1.2, p. 221 and the discussion on p. 224)

Ex 3: (Similar to #19, p. 225) Consider the linear operator

\[ L[y] = y^{(4)} - 10y'' + 9y \]

(a) Find \( L[e^{rt}] \).

(b) Determine four solutions of the DE: \( y^{(4)} - 10y'' + 9y = 0 \). Do you think the four solutions form a F.S.S.? Why?

Solutions: (a) \( L[e^{rt}] = (e^{rt})^{(4)} - 10(e^{rt})'' + 9e^{rt} \)

\[ = r^4e^{rt} - 10r^2e^{rt} + 9e^{rt} \]

\[ = (r^4 - 10r^2 + 9)e^{rt} \]

(b) If we let \( y = e^{rt} \) in \( y^{(4)} - 10y'' + 9y = 0 \) then by (a),
\[(r^4 - 10r^2 + 9)e^t = 0 \quad \text{so} \quad r^4 - 10r^2 + 9 = 0 \quad \text{or} \quad (r^2 - 1)(r^2 - 9) = 0\]

so \((r - 1)(r + 1)(r - 3)(r + 3) = 0 \quad \text{so} \quad r = 1, -1, 3, \text{ or } -3\). Thus, four solutions are

\[y_1(t) = e^t, \quad y_2(t) = e^{-t}, \quad y_3(t) = e^{3t}, \quad y_4(t) = e^{-3t}\]

Note: \[W(y_1, y_2, y_3, y_4)(t) = \begin{vmatrix} e^t & e^{-t} & e^{3t} & e^{-3t} \\ e^t & -e^{-t} & 3e^{3t} & -3e^{-3t} \\ e^t & e^{-t} & 9e^{3t} & 9e^{-3t} \\ e^t & -e^{-t} & 27e^{3t} & -27e^{-3t} \end{vmatrix} \]

\[= e^t \cdot e^{-t} \cdot e^{3t} \cdot e^{-3t} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & -3 \\ 1 & 1 & 9 & 9 \\ 1 & -1 & 27 & -27 \end{vmatrix} \]

\[= e^{t-3t+3t-3t} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & -3 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 24 & -24 \end{vmatrix} \]

\[= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 2 & -4 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & -48 \end{vmatrix} \]

\[= (1)(-2)(3)(-48) \]

\[= 768 \neq 0\]

Therefore \(y_1, y_2, y_3, y_4\) form a F.S.S. to \(y^{(4)} - 10y'' + 9y = 0\).

The general solution is \[y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{3t} + c_4 e^{-3t}\] where \(c_1, c_2, c_3, c_4\) are arbitrary constants.