Sec. 4.2 Homogeneous Equations with Constant Coefficients

HW p. 231: #11, 22, 29, 39  Due: Mon., Oct. 11

Schaum's: pp. 89-93

To solve \( n \text{th} \) order, linear, homogeneous DE with constant coefficients:

\[
(a) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0
\]

where the coefficients \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are real constants, we assume

\[ y = e^{rt} \]

This leads to

\[ a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \ldots + a_1 r e^{rt} + a_0 e^{rt} = 0 \]

and dividing through by \( e^{rt} \) yields

\[
(f) \quad a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 \quad \text{Characteristic Equation of (a).}
\]

Ex1 (13, p. 232) Find the general solution of \( 2y''' - 4y'' - 2y' + 4y = 0 \).

Sln: \( y = e^{rt} \) leads to

\[
\begin{align*}
4r^3 - 2r^2 - r + 4 & = 0 \\
\text{(1)} & = 0 \\
\text{(2)} & = 0
\end{align*}
\]

The possible rational zeros are of the form \( \frac{p}{q} \) where \( p \) divides 2 and \( q \) divides 1. Therefore \( \pm 1, \pm 2 \) are the possible rational zeros. We see that \( r = -1 \) is a zero by inspection, so \( r + 1 \) is a divisor of the polynomial.

\[
\begin{align*}
\frac{r^2 - 3r + 2}{r + 1} & = (r^2 - 3r + 2) \\
& = (r^2 + r^2) \\
& = (-3r^2 - r) \\
& = (-3r^2 - 3r) \\
& = 2r + 2 \\
& = 2r + 2 \\
& = 0 = R
\end{align*}
\]

Thus the characteristic equation is \( (r + 1)(r^2 - 3r + 2) = 0 \).
Factoring the quadratic yields \((r+1)(r-2)(r-1) = 0\) so \(r = -1, 2, \text{ or } 1\). Therefore \(y_1 = e^{-t}, y_2 = e^{2t}, y_3 = e^t\) are solutions to the DE. To check if they form a F.S.S., we compute their Wronskian:

\[
W(y_1, y_2, y_3)(t) = \begin{vmatrix}
-e^{-t} & e^{2t} & e^t \\
e^{-t} & 2e^{2t} & e^t \\
e^t & 4e^{2t} & e^t \\
\end{vmatrix}
\]

(factor \(e^t\) from column 1, \(e^{2t}\) from column 2, and \(e^t\) from column 3.)

\[
= \begin{vmatrix}
-t & 2t & t \\
e \cdot e \cdot e & -1 & 2 \\
e^t & 4 & 1 \\
\end{vmatrix}
\]

(Add row 1 to row 2, Add \(-1\) times row 1 to row 3, Expand by cofactors along row 3)

\[
= e^{2t} \begin{vmatrix}
1 & 1 & 1 \\
0 & 3 & 2 \\
0 & -3 & 0 \\
\end{vmatrix}
\]

(Expand by cofactors along row 3)

\[
= e^{2t} (-1)(3) \begin{vmatrix}
1 & 1 \\
0 & 2 \\
\end{vmatrix}
\]

\[
= -6e^{2t} \neq 0
\]

Therefore \(y_1 = e^{-t}, y_2 = e^{2t}, y_3 = e^t\) form a F.S.S. to the DE. The general solution is

\[
y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^t
\]

where \(c_1, c_2, \text{ and } c_3\) are arbitrary constants.
Ex 2 (Similar to #18, p. 232) Find the general solution of \( y^{(4)} - 4y''' + 10y'' - 12y' + 5y = 0 \)

**Sln:** \( y = e^t \) leads to \( r^4 - 4r^3 + 10r^2 - 12r + 5 = 0 \). The rational zeros are \( \frac{p}{q} \) where \( p \) divides 5 and \( q \) divides 1. Therefore \( \pm 1, \pm 5 \) are the candidates for rational zeros. By inspection we see that \( r = 1 \) is a zero. We use synthetic division to examine the multiplicity of \( r = 1 \) as a zero.

\[
\begin{array}{c|cccc}
1 & 1 & -4 & 10 & -12 & 5 \\
& & 1 & -3 & 7 & -5 \\
\hline
1 & 1 & -3 & 7 & -5 & 0 = R \\
& & 1 & -2 & 5 & 0 = R \\
\hline
& 1 & -2 & 5 & \frac{0}{0} = R
\end{array}
\]

\((r-1)(r^3 - 3r^2 + 7r - 5) = 0\)

\((r-1)(r^2 - 2r + 5) = 0\)

We use the quadratic formula to finish: \( r = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i \)

The roots of the characteristic equation are:

\( r = 1 \) (multiplicity two), \( r = 1 \pm 2i \).

Therefore \( y_1 = e^t, \ y_2 = te^t, \ y_3 = e^t \cos(2t), \ y_4 = e^t \sin(2t) \)

are solutions of the DE. One checks that

\[ W(y_1, y_2, y_3, y_4)(t) = 16e^{4t} \neq 0 \quad (cf. \#20(d), p. 225) \]

Therefore the set is a F.S.S. The general solution is

\[ y = c_1 e^t + c_2 te^t + e^t (c_3 \cos(2t) + c_4 \sin(2t)) \]

where \( c_1, c_2, c_3, c_4 \) are arbitrary constants.