Sec. 4.3: The Method of Undetermined Coefficients

HW p. 237: # 2, 11, 15

Schaum's: pp. 94-102 (esp. 11.7)

Due: Wed., Oct. 13

Sec. 4.4: Variation of Parameters

HW p. 242: #1 (on the interval \(-\pi/2 < t < \pi/2\)), 9, 13


In these two sections we will address the problem of solving the \(n\)th order linear nonhomogeneous equation

\[ y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y' + p_n(t)y = g(t). \]

We will concentrate mainly on the case when the coefficients are constant.

Ex 1 (#2, p. 237) Find the general solution of

\[ y^{(4)} - y = 3t + \cos(t). \]

Solution:

Step 1: \( y = e^{rt} \) in \( y^{(4)} - y = 0 \) leads to \( r^4 - 1 = 0 \). Factoring yields \((r^2 - 1)(r^2 + 1) = 0\) or \((r - 1)(r + 1)(r^2 + 1) = 0\). Therefore the characteristic equation has roots: \( r = 1, -1, i, -i \). The general solution of the associated homogeneous equation is \( y_c(t) = c_1e^t + c_2e^{-t} + c_3\cos(t) + c_4\sin(t) \).

Step 2: We will use the method of undetermined coefficients to find a particular solution of the nonhomogeneous equation. Since \( g(t) = 3t + \cos(t) \), we would normally use a trial form \( y_p(t) = At + B + C\cos(t) + D\sin(t) \). However, \( \cos(t) \) and \( \sin(t) \) are already solutions to the associated homogeneous
equation, so we must multiply $\cos(t) + D \sin(t)$ by $t$. (That is, $s=1$ in the table 3.5.1 on p. 181.) Therefore we use a trial form

$$y_p(t) = At + B + t(\cos(t) + D \sin(t))$$

where $A, B, C$, and $D$ are constants to be determined. Then

$$y_p' = A + \cos(t) + D \sin(t) + t(-\sin(t) + D \cos(t))$$

$$y_p'' = 2(-\sin(t) + D \cos(t)) + t(-\cos(t) - D \sin(t))$$

$$y_p''' = 3(-\cos(t) - D \sin(t)) + t(\sin(t) - D \cos(t))$$

$$y_p^{(4)} = 4(\sin(t) - D \cos(t)) + t(\cos(t) + D \sin(t))$$

$$y_p - (At + B)$$

We want

$$y_p^{(4)} - y_p = 3t + \cos(t)$$

so substituting expressions for $y_p^{(4)}$ and $y_p$ from above yields

$$-(At + B) + 4(C \sin(t) - D \cos(t)) = 3t + \cos(t)$$

or

$$-A - 4C \sin(t) - 4D \cos(t) = 3t + 0.1 + 0. \sin(t) + 1 \cos(t).$$

Equating like coefficients gives $-A = 3$, $-B = 0$, $4C = 0$, and $-4D = 1$.

Therefore $y_p(t) = -3t - \frac{1}{4} t \sin(t)$.

**Step 3**: Write the general solution $y = y_c + y_p$.

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t) - 3t - \frac{1}{4} t \sin(t)$$

where $c_1, c_2, c_3, c_4$ are arbitrary constants.
Variation of Parameters: A particular solution of

\[(*) \quad y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = g(t)\]

is

\[y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \ldots + u_n(t)y_n(t)\]

where \(y_1, y_2, \ldots, y_n\) is a fundamental set of solutions of the associated homogeneous equation of (*) and

\[u_i(t) = \int \frac{g(t)W_i(t)}{W(t)} \, dt \quad (i = 1, 2, \ldots, n).\]

Here \(W(t)\) denotes the Wronskian of \(y_1, y_2, \ldots, y_n\) at \(t\) and \(W_i(t)\) is the determinant obtained from \(W(t)\) by replacing the column containing \(y_i(t)\) and its derivatives by the column \(\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}\).

Notes:
1. If you want to use variation of parameters for \(y_p\), then you must remember to place the DE (*) in standard form. That is, the leading coefficient must be 1. (See #13 on p. 242.)
2. The variation of parameters formula is derived in the text on pp. 239-240.

Ex. 2 Find the general solution of

\[y''' + y' = \sec(t)\]

on the interval \(-\frac{\pi}{2} < t < \frac{\pi}{2}\.\]
Solution:

Step 1: $y = e^t$ in $y''' + y' = 0$ leads to $r^3 + r = 0$ or $r(r^2 + i) = 0$. Therefore the roots of the characteristic equation are $r = 0, i, -i$. Then

$y_1(t) = 1, y_2(t) = \cos(t), y_3(t) = \sin(t)$ are solutions of $y''' + y' = 0$.

Expand by cofactors along column 1,

$W(t) = \begin{vmatrix} i & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \cdot \begin{vmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{vmatrix} + 0 \begin{vmatrix} \cos(t) & \sin(t) \\ -\cos(t) & -\sin(t) \end{vmatrix} = 1$

Therefore $y_1, y_2, y_3$ is a F.S.S. for $y''' + y' = 0$.

Step 2: We use variation of parameters to find $y_p$ because $g(t) = \sec(t)$ is not an exponential, polynomial, or sine or cosine function (or sums or products of such).

Then $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t)$

Replace $[0]_1$ by this column,

$W_1(t) = \begin{vmatrix} 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \cdot \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$

Replace $[\cos(t), \sin(t), -\cos(t)]$ by this column,

$W_2(t) = \begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & \cos(t) & \sin(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \cdot \begin{vmatrix} \cos(t) & \sin(t) \\ 0 & \cos(t) \end{vmatrix} = -\cos(t)$

Replace $[-\sin(t), \cos(t), -\sin(t)]$ by this column,

$W_3(t) = \begin{vmatrix} 1 & \cos(t) & 0 \\ 0 & -\sin(t) & 0 \\ 0 & -\cos(t) & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -\sin(t) & 0 \\ -\cos(t) & 1 \end{vmatrix} = -\sin(t)$

$e^t u_1(t) = \int \frac{gW_1}{W} dt = \int \frac{\sec(t) \cdot 1}{\sec(t) + \tan(t)} dt = \ln|\sec(t) + \tan(t)| + C$

$e^t u_2(t) = \int \frac{gW_2}{W} dt = \int \frac{\sec(t)(-\cos(t))}{1} dt = \int -\cos(t) dt = -t + C$

$e^t u_3(t) = \int \frac{gW_3}{W} dt = \int \frac{-\sin(t)}{-\sin(t)} dt = \int 1 dt = t + C$
\[ u_3(t) = \int \frac{g W_3}{W} \, dt = \int \frac{\sec(t) (\sinh(t))}{1} \, dt = \int \frac{-\sinh(t) \, dt}{\cos(t)} \]

Let \( \alpha = \cos(t) \)

Then \( d\alpha = -\sin(t) \, dt \)

\[ = \int \frac{d\alpha}{\alpha} = \ln |\alpha| + C = \ln |\cos(t)| \]

Consequently, \[ y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 \]

\[ = \ln |\sec(t) + \tan(t)| - t \cos(t) + \sin(t) \ln |\cos(t)| \]

**Step 3:** Write the general solution \( y = y_c + y_p \):

\[ y(t) = c_1 + c_2 \cos(t) + c_3 \sin(t) + \ln |\sec(t) + \tan(t)| - t \cos(t) + \sin(t) \ln |\cos(t)| \]

where \( c_1, c_2, c_3 \) are arbitrary constants.