Chap. 6: The Laplace Transform

In this chapter we will solve IVP's by the Laplace transform method.

Sec. 6.1–6.2 → Step 1: Convert the IVP into an algebraic equation using the Laplace transform

Step 2: Solve the algebraic equation.

Sec. 6.2 → Step 3: Convert the solution of the algebraic equation into a solution of the IVP using the inverse Laplace transform.

Secs. 6.3–6.6 give more advanced properties of the Laplace transform that help us solve wider classes of IVP's, especially those where the driver changes abruptly.

Sec. 6.1 Definition of the Laplace Transform

HW p. 311: # 5, 11, 15, 27 Due: Wed., Oct. 20

Schaum's: pp. 211–223

Definition: Let f be a function defined on $[0,\infty)$. Then the Laplace transform of f at s is defined by

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

for those values of s for which the improper integral converges.

Ex 1 Find the Laplace transform of $f(t) = e^{4t}$.

Solution: $\mathcal{L}\{f\}(s) = \int_0^\infty e^{4t} \cdot e^{-st}\,dt = \lim_{M\to\infty} \int_0^M e^{(4-s)t}\,dt$

$$= \lim_{M\to\infty} \left.\frac{e^{(4-s)t}}{4-s}\right|_{t=0}^{M} = \lim_{M\to\infty} \frac{1}{4-s}\left(e^{(4-s)M} - 1\right) = \frac{-1}{4-s}$$ provided $4-s<0$; ie. $s>4$.

$$\therefore \boxed{\mathcal{L}\{e^{4t}\}(s) = \frac{1}{s-4} \text{ provided } s>4.}$$

You are to know by heart (for quizzes) and be able to derive (for exams) the following transform formulas.

Can be derived in a way similar to that for $\mathcal{L}\{e^{4t}\}(s)$

$f(t)$	$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$	
e^{at}	$\frac{1}{s-a}$	(provided $s > \mathrm{Re}(a)$)
$\cos(bt)$	$\frac{s}{s^2+b^2}$	(provided $s > 0$)
$\sin(bt)$	$\frac{b}{s^2+b^2}$	(provided $s > 0$)
t^n	$\frac{n!}{s^{n+1}}$	(provided $s > 0$)

We will often use the linearity property of the Laplace transform:

$$\mathcal{L}\{c_1f_1 + c_2f_2\}(s) = c_1\mathcal{L}\{f_1\}(s) + c_2\mathcal{L}\{f_2\}(s).$$

This follows directly from linearity of the integral and the definition of the Laplace transform as an integral transform:

$$\begin{aligned}\mathcal{L}\{c_1f_1+c_2f_2\}(s) &= \int_0^\infty \left[c_1f_1(t) + c_2f_2(t)\right]e^{-st}\,dt \\ &= c_1\int_0^\infty f_1(t)e^{-st}\,dt + c_2\int_0^\infty f_2(t)e^{-st}\,dt \\ &= c_1\mathcal{L}\{f_1\}(s) + c_2\mathcal{L}\{f_2\}(s).\end{aligned}$$

<u>Ex 2</u> Derive the Laplace transforms of the functions $f(t)=\cos(bt)$ and $g(t) = \sin(bt)$ where b is a real constant.

Using the first formula in the Laplace transform table with $a = ib$ we have

$$(*) \qquad \mathcal{L}\left\{ e^{ibt} \right\}(s) = \frac{1}{s-ib} \qquad \text{provided } s > \operatorname{Re}(ib) = 0.$$

But the Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and the linearity property of Laplace transforms gives

$$(**) \quad \mathcal{L}\left\{ e^{ibt} \right\}(s) = \mathcal{L}\left\{ \cos(bt) + i\sin(bt) \right\}(s) = \mathcal{L}\left\{ \cos(bt) \right\}(s) + i\,\mathcal{L}\left\{ \sin(bt) \right\}(s)$$

Equating the two expressions in $(*)$ and $(**)$ gives

$$\mathcal{L}\{\cos(bt)\}(s) + i\mathcal{L}\{\sin(bt)\}(s) = \frac{1}{s-ib} \cdot \left(\frac{s+ib}{s+ib} \right) = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}.$$

Equating real parts and imaginary parts on the left and right sides yields

$$\mathcal{L}\left\{ \cos(bt) \right\}(s) = \frac{s}{s^2+b^2} \qquad \text{and} \qquad \mathcal{L}\{\sin(bt)\}(s) = \frac{b}{s^2+b^2}$$

for $s > 0$.

<u>Ex 3</u> (similar to ex. 8, p. 310) Find the Laplace transform of

$$f(t) = 2e^{3t} + 4\cos(5t) - 3\sin(2t).$$

Solution: Using linearity of Laplace transforms and the first three formulas in the table of Laplace transforms yields

$$\mathcal{L}\{f\}(s) = \mathcal{L}\left\{ 2e^{3t} + 4\cos(5t) - 3\sin(2t) \right\}(s)$$

$$= 2\mathcal{L}\left\{ e^{3t} \right\}(s) + 4\mathcal{L}\left\{ \cos(5t) \right\}(s) - 3\mathcal{L}\left\{ \sin(2t) \right\}(s)$$

$$\mathcal{L}\{f\}(s) = \frac{2}{s-3} + \frac{4s}{s^2+25} - \frac{6}{s^2+4}\,. \qquad (\text{This is valid provided } s>3.)$$

To derive the Laplace transform of t^n, we will employ the Gamma function (see #26, 27 pp. 311–312).

<u>Definition</u>: $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\,dx$ for $p > 0$.

<u>Examples</u>:

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\,dx = \lim_{M\to\infty} \int_0^M e^{-x}\,dx = \lim_{M\to\infty}\left(-e^{-x}\Big|_0^M\right) = \lim_{M\to\infty}\left(1-e^{-M}\right) = 1.$$

$$\Gamma(1/2) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx = \int_{x=0}^\infty x^{-1/2} e^{-x}\,dx = \int_{w=0}^\infty \cancel{w^{-1}}\, e^{-w^2}\, 2\cancel{w}\,dw = 2\int_0^\infty e^{-w^2}\,dw = \sqrt{\pi}.$$

Let $x = w^2$.
Then $dx = 2w\,dw$

(Derive $\int_{-\infty}^\infty e^{-w^2}\,dw = \sqrt{\pi}$ if students don't know it.)

<u>FACT</u>: $\Gamma(p+1) = p\Gamma(p)$ if $p>0$.

<u>Reason</u>:
$$\Gamma(p+1) = \int_0^\infty x^{(p+1)-1} e^{-x}\,dx = \int_0^\infty \underbrace{x^p}_{v}\, \underbrace{e^{-x}\,dx}_{dV} = \cancel{-x^p e^{-x}\Big|_0^\infty} - \int_0^\infty -e^{-x}\, p x^{p-1}\,dx$$

$$= p\int_0^\infty x^{p-1} e^{-x}\,dx = p\,\Gamma(p).$$

<u>Examples</u>: $\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$

$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2$

$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2$

$\vdots$

$\Gamma(n+1) = n\Gamma(n) = n!$ ($n = 1, 2, 3, \ldots$ It also suggests $0! = \Gamma(1) = 1$ is a natural definition.)

Examples (cont.): $\Gamma(3/2) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

$$\Gamma(5/2) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3\sqrt{\pi}}{2\cdot 2}$$

$$\Gamma(7/2) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5\cdot 3\sqrt{\pi}}{2\cdot 2\cdot 2}$$

$$\vdots$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\cdot\sqrt{\pi}}{2^n} \qquad (n = 1, 2, 3, \ldots)$$

Fact: (HW #27, p. 312) $\mathcal{L}\{t^p\}(s) = \dfrac{\Gamma(p+1)}{s^{p+1}}$ for $s > 0$ and $p > -1$.

The special case when p is a nonnegative integer, say $p = n$, is important for us:

$$\boxed{\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}\ .$$

Ex 4 If $f(t) = t^2 + 5 - e^{-9t}$, find $\mathcal{L}\{f\}(s)$.

Solution: Using linearity of the Laplace transform and the first and fourth entries in the table of Laplace transforms yields

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{t^2 + 5 - e^{-9t}\}(s) = \mathcal{L}\{t^2\}(s) + 5\mathcal{L}\{t^0\}(s) - \mathcal{L}\{e^{-9t}\}(s)$$

$$= \boxed{\frac{2}{s^3} + \frac{5}{s} - \frac{1}{s+9}} \qquad (\text{provided } s > 0)$$

(Omit if pressed for time.)

Some functions are "too rough" to possess a Laplace transform. For instance, if

$$f(t) = \begin{cases} 1 & \text{if } t \text{ is rational,} \\ 0 & \text{if } t \text{ is irrational,} \end{cases}$$

then $\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$ does not exist for <u>any</u> real number s.

On the other hand, some functions "grow too fast" to possess a Laplace transform. For example,

$$\mathcal{L}\{e^{t^2}\}(s) = \int_0^\infty e^{t^2} \cdot e^{-st}\,dt = \int_0^\infty e^{\overbrace{t(t-s)}^{\text{positive if } t>s}}\,dt = +\infty \quad \text{for } \underline{\text{all}} \text{ real } s.$$

The existence theorem (see below) says that if f is "smooth enough" and "doesn't grow too fast" then the Laplace transform of f exists for s sufficiently large.

<u>Theorem 6.1.2</u> (p. 308): Let f be a real-valued function defined on $0 \le t < \infty$ and possessing the following properties:

f is "smooth enough" → (1) for every $A > 0$, f is piecewise continuous on the interval $0 \le t \le A$;

f "doesn't grow too quickly" → (2) there exist real numbers K, a, and M such that $|f(t)| \le Ke^{at}$ for all $t \ge M$.

Then the Laplace transform of f at s,

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

exists for all $s > a$.

<u>Note</u>: A function f satisfying condition (2) of the above theorem is said to be of <u>exponential order</u> as $t \to \infty$.