Sec. 6.2 Solution of Initial Value Problems

HW p. 320: # 3, 8 Due: Fri., Oct. 22

11, 19 Due: Mon., Oct. 25

Schaum's: pp. 224–232 and pp. 242–248

Definition: If $\mathcal{L}\{f\}(s) = F(s)$ then we say that $f = f(t)$ is an inverse Laplace transform of $F = F(s)$, and we write $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Examples: $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ because $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$.

$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$ because $\mathcal{L}\{e^{2t}\}(s) = \frac{1}{s-2}$.

$\mathcal{L}^{-1}\{F(s)\}$	$F(s)$
e^{at}	$\frac{1}{s-a}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\frac{\sin(bt)}{b}$	$\frac{1}{s^2+b^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$

KNOW THIS TABLE BY HEART (for quizzes)

Elementary Properties of Inverse Laplace Transforms

$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1\mathcal{L}^{-1}\{F_1(s)\} + c_2\mathcal{L}^{-1}\{F_2(s)\}$$

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}\mathcal{L}^{-1}\{F(s)\}$$

Note: This is equivalent to saying that if $\mathcal{L}\{f\}(s) = F(s)$ then $\mathcal{L}\{e^{ct}f(t)\}(s) = F(s-c)$.

(See Theorem 6.3.2, p. 328)

Proof of second elementary property: Suppose that $\mathcal{L}\{f(t)\} = F(s)$. Then $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{ct}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-c)t}\,dt = F(s-c)$.

Ex 1 (#4, p. 320) Find $\mathcal{L}^{-1}\left\{\frac{3s}{s^2-s-6}\right\}$.

Solution: By the partial fraction decomposition theorem,

$$\frac{3s}{s^2-s-6} = \underbrace{\frac{3s}{(s-3)(s+2)}}_{\text{Distinct linear factors}} = \frac{A}{s-3} + \frac{B}{s+2}$$

for some constants A and B. Multiplying through by $(s-3)(s+2)$ yields

$$3s = \left(\frac{A}{s-3} + \frac{B}{s+2}\right)(s-3)(s+2) = A(s+2) + B(s-3).$$

To find A, set $s=3$: $3(3) = A(3+2) + B(3-3) \Rightarrow A = 9/5$.

To find B, set $s=-2$: $3(-2) = A(-2+2) + B(-2-3) \Rightarrow B = 6/5$.

Therefore $$\mathcal{L}^{-1}\left\{\frac{3s}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{9/5}{s-3} + \frac{6/5}{s+2}\right\}$$

$$= \frac{9}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \boxed{\frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}}.$$

Ex 2 (Similar to #8, p. 320) Find $\mathcal{L}^{-1}\left\{\frac{s+2}{s^4+s^2}\right\}$.

Solution: By the partial fraction decomposition theorem,

$$\frac{s+2}{s^4+s^2} = \frac{s+2}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

Repeated linear factors + Irreducible quadratic factor

where A, B, C, and D are constants. Multiplying through by $s^2(s^2+1)$ yields

$$s+2 = \left(\frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}\right)s^2(s^2+1) = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

To find B, set $s=0$: $0+2 = A(0) + B(0+1) + (C0+D)0 \Rightarrow B=2$.

To find C and D, set $s=i$: $i+2 = A(0) + B(0) + (Ci+D)(-1) = -Ci - D$

$$\Rightarrow 1=-C \text{ and } 2=-D.$$

To find A, we can set s equal to any number but $1, i,$ and i. We set $s=1$:

$$1+2 = A(2) + B(2) + C+D$$

$$3 = 2A + 4 - 1 - 2 \qquad \Rightarrow \quad A = 1$$

Therefore
$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^4+s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s^2} + \frac{-s-2}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 1 - 2t - \cos(t) - 2\sin(t).$$

Omit if pressed for time

<u>Ex 3</u> (#9, p. 320) Find $\mathcal{L}^{-1}\left\{\frac{1-2s}{s^2+4s+5}\right\}$.

<u>Solution</u>: Note that s^2+4s+5 is an irreducible quadratic factor so $\frac{1-2s}{s^2+4s+5}$ is already in partial fraction decomposed form: $\frac{As+B}{s^2+4s+5}$ where $A=-2$ and $B=1$. We complete the square in the denominator and use properties of inverse Laplace transforms.

$$\mathcal{L}^{-1}\left\{\frac{1-2s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1-2(s+2)+4}{s^2+4s+4+1}\right\} = \mathcal{L}^{-1}\left\{\frac{-2(s+2)+5}{(s+2)^2+1}\right\}$$

$$= -2\mathcal{L}^{-1}\Bigg\{\underbrace{\frac{s+2}{(s+2)^2+1}}_{F(s+2) \text{ where } F(s)=\frac{s}{s^2+1}}\Bigg\} + 5\mathcal{L}^{-1}\Bigg\{\underbrace{\frac{1}{(s+2)^2+1}}_{G(s+2) \text{ where } G(s)=\frac{1}{s^2+1}}\Bigg\}$$

$$= -2e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 5e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \boxed{-2e^{-2t}\cos(t) + 5e^{-2t}\sin(t)}$$

~ To here Day 1 ~

To solve differential equations using Laplace transforms, we will need know how to transform derivatives. This is contained in:

Don't write this on the board. Have students read along in their texts.

Theorem 6.2.1 (p.313): Let f be continuous and f' piecewise continuous on every closed, bounded interval $0 \le t \le A$. If there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for all $t \ge M$, then $\mathcal{L}\{f'\}(s)$ exists for all $s > a$ and $\boxed{\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).}$

(See p. 313 in the textbook for a proof.)

Notes: If f' and f'' satisfy the conditions imposed on f and f', respectively, in Theorem 6.2.1 then

$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0)$$

$$= s\left(s\mathcal{L}\{f\}(s) - f(0)\right) - f'(0)$$

$$\boxed{\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)}$$

Ex 4 (#14, p.320) Use the Laplace transform to solve the IVP

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: Step 1: We convert the IVP into an algebraic equation using the Laplace transf.

$$\mathcal{L}\{y'' - 4y' + 4y\}(s) = \mathcal{L}\{0\}(s)$$

$$\mathcal{L}\{y''\}(s) - 4\mathcal{L}\{y'\}(s) + 4\mathcal{L}\{y\}(s) = 0$$

$$s^2\underline{\mathcal{L}\{y\}(s)} - s\cancel{y(0)}^{\,1} - \cancel{y'(0)}^{\,1} - 4\left(s\underline{\mathcal{L}\{y\}(s)} - \cancel{y(0)}^{\,1}\right) + 4\underline{\mathcal{L}\{y\}(s)} = 0$$

The underlined $\mathcal{L}\{y\}(s)$ is the unknown (function).

Step 2: We solve the algebraic equation.

$$(s^2 - 4s + 4)\mathcal{L}\{y\}(s) = s - 3$$

$$\mathcal{L}\{y\}(s) = \frac{s-3}{s^2-4s+4}$$

Step 3: We convert the solution of the algebraic equation into a solution of the IVP using the inverse Laplace transform.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s-3}{s^2-4s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2-1}{(s-2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \overbrace{\frac{1}{(s-2)^2}}^{F(s-2) \text{ where } F(s)=\frac{1}{s^2}}\right\}$$

$$= e^{2t} - e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{y(t) = e^{2t} - te^{2t}}$$

Ex 5 (#18, p. 320) Use the Laplace transform to solve the IVP

$$y^{(4)} - y = 0, \quad y(0) = 1, \; y'(0) = 0, \; y''(0) = 1, \; y'''(0) = 0.$$

<u>Solution</u>: <u>Step 1</u>: $\mathcal{L}\{y^{(4)} - y\}(s) = \mathcal{L}\{0\}(s)$

$$s^4\mathcal{L}\{y\}(s) - s^3\cancelto{1}{y(0)} - s^2\cancelto{0}{y'(0)} - s\cancelto{1}{y''(0)} - \cancelto{0}{y'''(0)} - \mathcal{L}\{y\}(s) = 0$$

<u>Step 2</u>: $(s^4 - 1)\mathcal{L}\{y\}(s) = s^3 + s$

$$\mathcal{L}\{y\}(s) = \frac{s(s^2+1)}{s^4-1}$$

<u>Step 3</u>: $y(t) = \mathcal{L}^{-1}\left\{\dfrac{s(s^2+1)}{s^4-1}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s\cancel{(s^2+1)}}{\cancel{(s^2+1)}(s^2-1)}\right\}$

$$\frac{s}{s^2-1} = \frac{s}{(s-1)(s+1)} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow s = A(s-1) + B(s+1)$$

To find A, set $s = -1$: $-1 = A(-1-1) + B(-1+1) \Rightarrow A = 1/2$

To find B, set $s = 1$: $1 = A(1-1) + B(1+1) \Rightarrow B = 1/2$.

$$\therefore\ y(t) = \mathcal{L}^{-1}\left\{\frac{1/2}{s+1} + \frac{1/2}{s-1}\right\} = \boxed{\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh(t)}$$