Sec. 6.5: Impulse Functions

Homework p. 343: #1, 4, 9
Schaum's: ??

Consider \( my'' + Yy' + ky = g(t) \). In this section we learn to model a sudden "blow" or "shock" with appropriate \( g(t) \).

We begin with a definition: The integral

\[
I_g(\tau) = \int_{-\tau}^{\tau} g(t) \, dt
\]

is called the impulse of the force \( g \) over the time interval \(-\tau < t < \tau\).

For example, for each positive integer \( n = 1, 2, 3, \ldots \) consider the force defined by

\[
g_n(t) = \begin{cases} \frac{n}{2} & \text{if } -\frac{1}{n} < t < \frac{1}{n}, \\ 0 & \text{otherwise}. \end{cases}
\]

Some terms in this sequence of forces have their graphs displayed below. Note that they "peak up" near zero.
Let us calculate the impulse of $g_n$ on its "support", the interval $-\frac{1}{n} < t < \frac{1}{n}$.

$$I_{g_n}(\frac{1}{n}) = \int_{-\frac{1}{n}}^{\frac{1}{n}} g_n(t) \, dt$$

$$= \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} \, dt$$

$$= 1.$$

Therefore $\{g_n\}_{n=1}^{\infty}$ is a sequence of unit impulses that are "supported" in smaller and smaller neighborhoods of zero. We idealize this sequence of unit impulses $\{g_n\}_{n=1}^{\infty}$ by "taking the limit" as $n \to \infty$ in the following way.

**Definition:** For each continuous function $f$ on $(-\infty, \infty)$ define

$$\text{(\#)} \quad \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) g_n(t) \, dt.$$

Notes: The "limit" of the $g_n$'s is denoted by $\delta$ and is called the **unit impulse** or the **Dirac delta** (after the physicist P.M. Dirac who introduced it). The symbol $\delta$ is completely defined by the property $(\#)$. Contrary to the terminology in Boyce and DiPrima, $\delta$ is not a function at all. Rather it is a "distribution" or "generalized function".

The property $(\#)$ is a little cumbersome to use very effectively. Therefore
let's examine the limit of integrals in (*) more closely. observe that

\[
\int_{-\infty}^{\infty} f(t)g_n(t)\,dt = \int_{-\infty}^{-\frac{1}{n}} f(t)g_n(t)\,dt + \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)g_n(t)\,dt + \int_{\frac{1}{n}}^{\infty} f(t)g_n(t)\,dt
\]

\[
= \int_{-\infty}^{\frac{1}{n}} \frac{f(t)}{2}\,dt
\]

\[
= \frac{1}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)\,dt . \quad (\dag)
\]

Recall that the average value of a function g on the interval \(a \leq x \leq b\) is given by

\[
\bar{g} = \frac{1}{b-a} \int_{a}^{b} g(x)\,dx .
\]

Geometrically, \(\bar{g}\) is the height of the rectangle, with base \(a \leq x \leq b\) on the x-axis, whose area is the same as the area under the curve \(y = g(x)\) over the interval \(a \leq x \leq b\).

Returning to (\dag), we see that \(\frac{1}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)\,dt\) represents the average value of the function \(f\) over the interval \(-\frac{1}{n} < t < \frac{1}{n}\). (See the accompanying graph on the next page.)
\[
y = f(t)
\]
\[
\bar{f} = \frac{1}{\frac{2}{n} - \frac{1}{n}} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t) \, dt
\]

Since \( f \) is continuous at 0 by assumption, it is clear geometrically (and can be proven analytically) that as \( n \to \infty \), the average value
\[
\frac{1}{\frac{2}{n} - \frac{1}{n}} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t) \, dt
\]
approaches \( f(0) \).

Therefore, the definition of the Dirac delta can be elaborated as follows.

**Definition:** For each bounded continuous function \( f \) on \((-\infty, \infty)\) define

\[
(*) \quad \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) \varphi_{1/n}(t) \, dt = \lim_{n \to \infty} \frac{1}{\frac{2}{n} - \frac{1}{n}} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t) \, dt = f(0).
\]

This is a very important and useful way to define the Dirac delta. Note that (*) says the action of \( \delta \) on any bounded continuous function \( f \)

is to "sift out" the number \( f(0) \) from \( f \). Consequently (*)

is referred to as the "sifting property" of the Dirac delta.

More generally, if \( t_0 \) is any real number, then the translate \( \delta(t-t_0) \)
of the Dirac delta by \( t_0 \) is defined by the property

\[
(**) \quad \int_{-\infty}^{\infty} f(t) \delta(t-t_0) \, dt = f(t_0)
\]

for every bounded continuous function \( f \) on \((-\infty, \infty)\).

**Q:** What is the Laplace transform of \( \delta(t-t_0) \) for \( t_0 \geq 0 \)?
A: 
\[ \mathcal{L} \{ \delta(t-t_0) \} (s) = \int_0^\infty \delta(t-t_0) e^{-st} \, dt = e^{-st_0} \] by the sifting property (\#4).

In particular, taking \( t_0 = 0 \) we have

\[ \mathcal{L} \{ \delta(t) \} (s) = 1. \]

These formulas allow us to handle "shocks" and "blows" in our forcing "functions" \( g(t) \) using the Laplace transform.

**Ex. 1:** (Similar to \#1, p. 343) Find the solution of the IVP

\[ 2y'' + 4y' + 10y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0, \]

and sketch the graph of the solution on the interval \( 0 \leq t \leq 10 \).

Preliminary discussion of the physical interpretation of this problem: The IVP (\#) models the motion of a body of mass 2 attached to a spring with stiffness coefficient 10 in the presence of a medium whose damping constant is 4. The meaning of the forcing term \( \delta(t-5) \) is that a sudden external unit impulse acts on the body at the time \( t = 5 \). Since the initial conditions are homogeneous we expect no motion in the body over the time interval \( 0 \leq t < 5 \). We expect the body to move (downward) in response to the impulse at \( t = 5 \) and then oscillate in a damped mode thereafter. Consequently, we expect the graph of the solution over \( 0 \leq t \leq 10 \) to look something like this:
Let's see what the quantitative details of the solution to \((*)\) are.

**Solution:** We use the Laplace transform method to solve \((*)\) because of the presence of the translate of the Dirac delta in the forcing term. Taking the Laplace transform of the DE in \((*)\) yields

\[
\mathcal{L}\{2y'' + 4y' + 10y\}(s) = \mathcal{L}\{\delta(t-5)\}(s)
\]

\[
2(s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)) + 4(s\mathcal{L}\{y\}(s) - y(0)) + 10\mathcal{L}\{y\}(s) = e^{-5s}
\]

\[
(2s^2 + 4s + 10)\mathcal{L}\{y\}(s) = e^{-5s}
\]

\[
\mathcal{L}\{y\}(s) = \frac{e^{-5s}}{2(s^2 + 2s + 5)}.
\]

Taking the inverse Laplace transform gives

\[
y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{2(s^2 + 2s + 5)}\right\}
\]

\[
= u_5(t)f(t-5)
\]

where \(f(t) = \mathcal{L}^{-1}\left\{\frac{1}{2(s^2 + 2s + 5)}\right\} = \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2 + 4}\right\} = \frac{1}{4} e^{-t} \sin(2t).
\]
Thus, \[ y(t) = u_5(t) \cdot \frac{1}{4} e^{-(t-5)} \sin(2(t-5)) \]. For graphing purposes, we rewrite the solution as follows:

\[
y(t) = \begin{cases} 
0 & \text{if } 0 \leq t < 5, \\
\frac{1}{4} e^{-(t-5)} \sin(2(t-5)) & \text{if } t \geq 5.
\end{cases}
\]

The graph is:

```
| y | 0.1 |
---|-----|
| o  |     |
```

\[ y = y(t) = \frac{1}{4} e^{-(t-5)} \sin(2(t-5)) \cdot u(t) \]

Response to a unit impulse at \( t=5 \)
to a mass-spring system with \( m=2, \gamma=4 \), and \( k=10 \).

**Ex 2** (Similar to #9, p. 343) Solve the IVP

\[ y'' + y = \delta(t - \frac{\pi}{2}) + \delta(t - 2\pi), \quad y(0)=1, \ y'(0)=0, \]

and sketch the graph of the solution on \( 0 \leq t \leq 4\pi \).

Solution: We use the Laplace transform method because of the presence of Dirac deltas in the forcing term. We take the Laplace transform
of the DE:

\[
L\{y'' + y\}(s) = L\{s(t-\pi/2) + s(t-2\pi)\}(s)
\]

\[
s^2L\{y\}(s) - sy(0) - y'(0) + L\{y\}(s) = e^{-\pi s} + e^{-2\pi s}
\]

\[
(s^2 + 1)L\{y\}(s) = s + e^{-\pi s} + e^{-2\pi s}
\]

\[
L\{y\}(s) = \frac{s}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1} + \frac{e^{-2\pi s}}{s^2 + 1}
\]

Taking the inverse Laplace transform yields

\[
y(t) = L^{-1}\left\{\frac{s}{s^2 + 1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 1}\right\} + L^{-1}\left\{\frac{e^{-2\pi s}}{s^2 + 1}\right\}
\]

\[
y(t) = \cos(t) + u_{\pi/2}(t)f(t-\pi) + u_{2\pi}(t)f(t-2\pi)
\]

where \(f(t) = L^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin(t)\). Therefore,

\[
y(t) = \cos(t) + u_{\pi/2}(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi)
\]

But \(\sin(t-\pi/2) = \sin(t)\cos(\pi/2) - \cos(t)\sin(\pi/2) = -\cos(t)\) and \(\sin(t-2\pi) = \sin(t)\)

so

\[
y(t) = \cos(t) - u_{\pi/2}(t)\cos(t) + u_{2\pi}(t)\sin(t)
\]

To sketch the graph we rewrite the solution:
\[ y(t) = \begin{cases} 
\cos(t) & \text{if } 0 \leq t < \frac{\pi}{2}, \\
\cos(t) - \cos(t) & \text{if } \frac{\pi}{2} \leq t < 2\pi, \\
\cos(t) - \cos(t) + \sin(t) & \text{if } 2\pi \leq t < \infty, 
\end{cases} \]

\[ = \begin{cases} 
\cos(t) & \text{if } 0 \leq t < \frac{\pi}{2}, \\
0 & \text{if } \frac{\pi}{2} \leq t < 2\pi, \\
\sin(t) & \text{if } 2\pi \leq t < \infty. 
\end{cases} \]

The graph of the solution follows: