Sec. 6.5 : Impulse Functions

HW p. 343 : #1, 4, 9   Due: Mon., Nov. 1

Schaum's : ??

Consider $my'' + \gamma y' + ky = g(t)$. In this section we learn to model a sudden "blow" or "shock" with appropriate $g(t)$.

We begin with a definition: The integral

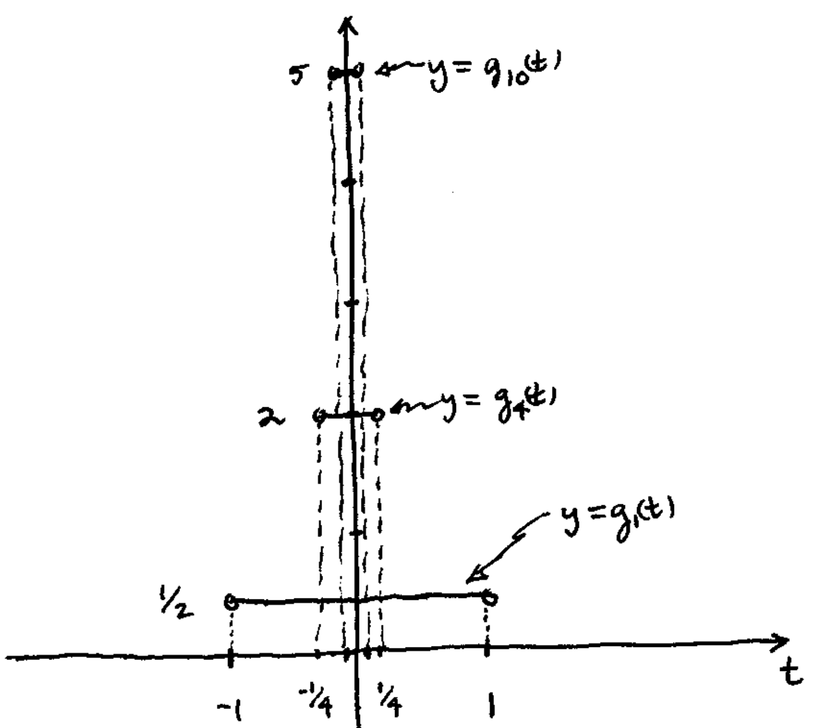
$$I_g(\tau) = \int_{-\tau}^{\tau} g(t)\,dt$$

is called the <u>impulse</u> of the force g over the time interval $-\tau < t < \tau$.

For example, for each positive integer $n = 1, 2, 3, \ldots$ consider the force defined by

$$g_n(t) = \begin{cases} \frac{n}{2} & \text{if } -\frac{1}{n} < t < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Some terms in this sequence of forces have their graphs displayed below. Note that they "peak up" near zero.



Let us calculate the impluse of g_n on its "support", the interval $-\frac{1}{n} < t < \frac{1}{n}$.

$$I_{g_n}\left(\tfrac{1}{n}\right) = \int_{-1/n}^{1/n} g_n(t)\,dt$$

$$= \int_{-1/n}^{1/n} \frac{n}{2}\,dt$$

$$= 1 .$$

Therefore $\{g_n\}_{n=1}^{\infty}$ is a sequence of unit impulses that are "supported" in smaller and smaller neighborhoods of zero. We idealize this sequence of unit impulses $\{g_n\}_{n=1}^{\infty}$ by "taking the limit" as $n \to \infty$ in the following way.

<u>Definition</u>: For each bounded continuous function f on $(-\infty,\infty)$ define

$$(*) \qquad \int_{-\infty}^{\infty} f(t)\delta(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) g_n(t)\,dt .$$

<u>Notes</u>: The "limit" of the g_n's is denoted by δ and is called the <u>unit impulse</u> or the <u>Dirac delta</u> (after the physicist P.M. Dirac who introduced it). The symbol δ is completely defined by the property $(*)$. Contrary to the terminology in Boyce and DiPrima, δ is not a function at all. Rather it is a "distribution" or "generalized function".

The property $(*)$ is a little cumbersome to use very effectively. Therefore let's examine the limit of integrals in (*) more closely. observe that

$$\int_{-\infty}^{\infty} f(t)g_n(t)\,dt = \int_{-\infty}^{-1/n} f(t)\underbrace{g_n(t)}_{0 \text{ for } t<-\frac{1}{n}}dt + \int_{-1/n}^{1/n} f(t)g_n(t)\,dt + \int_{1/n}^{\infty} f(t)\underbrace{g_n(t)}_{0 \text{ for } t>1/n}dt$$

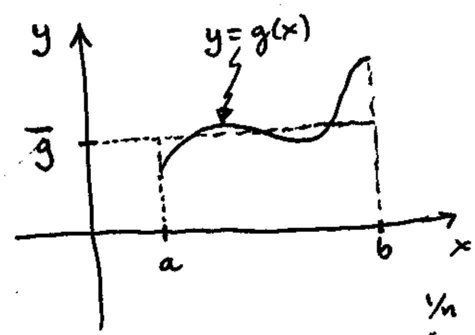
$$= \int_{-1/n}^{1/n} f(t)\frac{n}{2}\,dt$$

$$= \frac{1}{\frac{2}{n}}\int_{-1/n}^{1/n} f(t)\,dt \,. \qquad (\dagger)$$

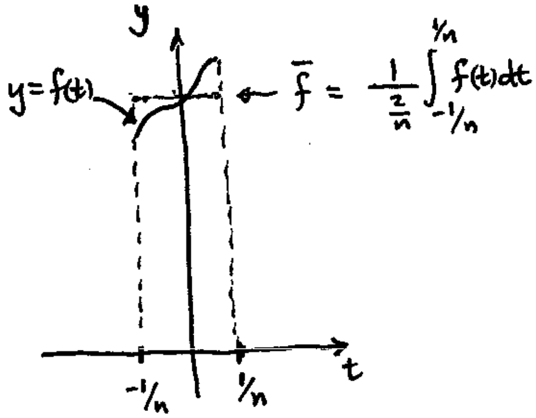
Recall that the average value of a function g on the interval $a \le x \le b$ is given by

$$\bar{g} = \frac{1}{b-a}\int_a^b g(x)\,dx \,.$$

Geometrically, $\bar{g}$ is the height of the rectangle, with base $a \le x \le b$ on the x-axis, whose area is the same as the area under the curve $y = g(x)$ over the interval $a \le x \le b$.



Returning to $(\dagger)$, we see that $\frac{1}{\frac{2}{n}}\int_{-1/n}^{1/n} f(t)\,dt$ represents the average value of the function f over the interval $-\frac{1}{n} < t < \frac{1}{n}$ (See the accompanying graph on the next page.)



Since f is continuous at 0 by assumption, it is clear geometrically (and can be proven analytically) that as $n \to \infty$, the average value $\frac{1}{\frac{2}{n}}\int_{-1/n}^{1/n} f(t)dt$ approaches $f(0)$.

Therefore, the definition of the Dirac delta can be elaborated as follows.

<u>Definition</u>: For each bounded continuous function f on $(-\infty,\infty)$ define

$$(*) \qquad \int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)g_n(t)dt = \lim_{n\to\infty}\frac{1}{2/n}\int_{-1/n}^{1/n} f(t)dt = f(0).$$

This is a very important and useful way to define the Dirac delta. Note that $(*)$ says the action of δ on any bounded continuous function f is to "sift out" the number $f(0)$ from f. Consequently $(*)$ is referred to as the "sifting property" of the Dirac delta.

More generally, if t_0 is any real number, then the translate $\delta(t-t_0)$ of the Dirac delta by t_0 is defined by the property

$$(**) \qquad \int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)dt = f(t_0)$$

for every bounded continuous function f on $(-\infty,\infty)$.

Q: What is the Laplace transform of $\delta(t-t_0)$ for $t_0 \geq 0$?

A: $\boxed{\mathcal{L}\{\delta(t-t_0)\}(s) = \int_0^\infty \delta(t-t_0)e^{-st}\,dt = e^{-st_0}}$ by the

sifting property (**).

In particular, taking $t_0 = 0$ we have

$$\boxed{\mathcal{L}\{\delta(t)\}(s) = 1.}$$

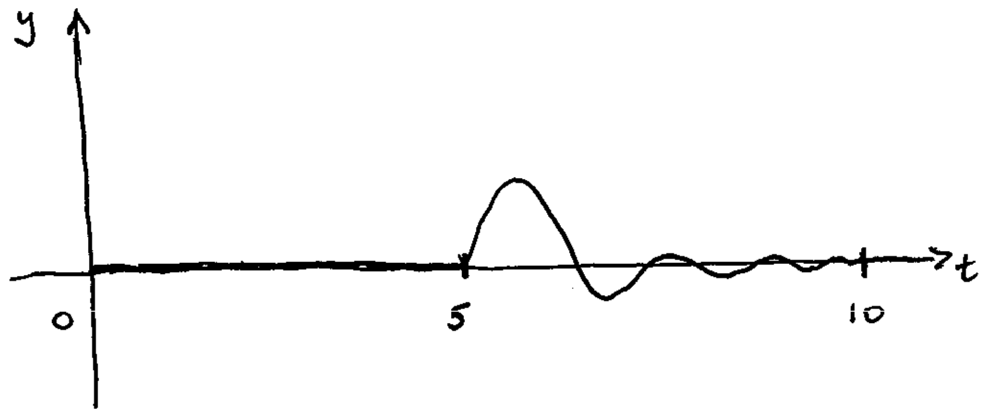
These formulas allow us to handle "shocks" and "blows" in our forcing "functions" $g(t)$ using the Laplace transform.

<u>Ex 1</u> (Similar to #1, p. 343) Find the solution of the IVP

(*) $\qquad 2y'' + 4y' + 10y = \delta(t-5), \quad y(0) = 0, \; y'(0) = 0,$

and sketch the graph of the solution on the interval $0 \le t \le 10$.

<u>Preliminary discussion of the physical interpretation of this problem</u>: The IVP (*) models the motion of a body of mass 2 attached to a spring with stiffness coefficient 10 in the presence of a medium whose damping constant is 4. The meaning of the forcing term $\delta(t-5)$ is that a sudden external unit impulse acts on the body at the time $t = 5$. Since the initial conditions are homogeneous we expect no motion in the body over the time interval $0 \le t < 5$. We expect the body to move (downward) in response to the impulse at $t = 5$ and then oscillate in a damped mode thereafter. Consequently, we expect the graph of the solution over $0 \le t \le 10$ to look something like this:



Let's see what the quantitative details of the solution to (*) are.

Solution: We use the Laplace transform method to solve (*) because of the presence of the translate of the Dirac delta in the forcing term. Taking the Laplace transform of the DE in (*) yields

$$\mathcal{L}\{2y'' + 4y' + 10y\}(s) = \mathcal{L}\{\delta(t-5)\}(s)$$

$$2\left(s^2\mathcal{L}\{y\}(s) - s\cancelto{0}{y(0)} - \cancelto{0}{y'(0)}\right) + 4\left(s\mathcal{L}\{y\}(s) - \cancelto{0}{y(0)}\right) + 10\mathcal{L}\{y\}(s) = e^{-5s}$$

$$(2s^2 + 4s + 10)\mathcal{L}\{y\}(s) = e^{-5s}$$

$$\mathcal{L}\{y\}(s) = \frac{e^{-5s}}{2(s^2 + 2s + 5)}.$$

Taking the inverse Laplace transform gives

$$y(t) = \mathcal{L}^{-1}\left\{e^{-5s} \cdot \frac{1}{2(s^2+2s+5)}\right\}$$

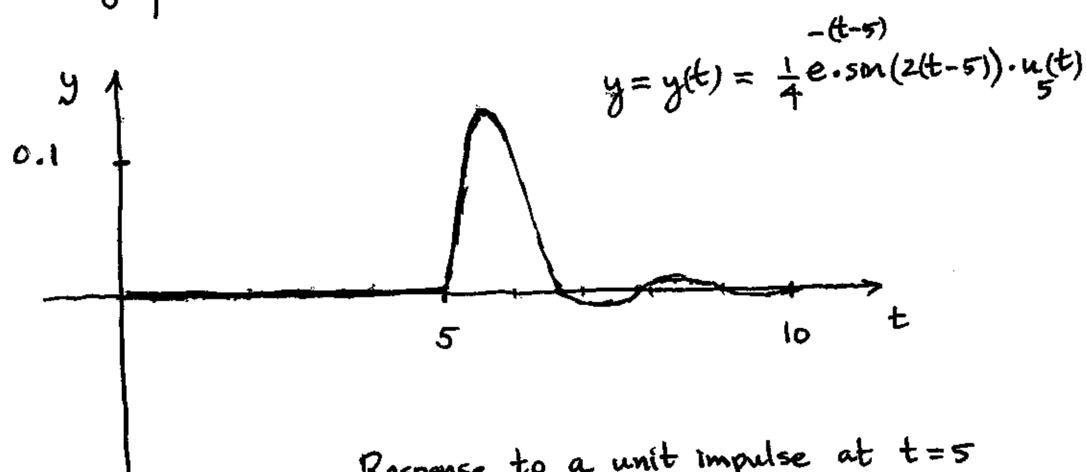
$$= u_5(t) f(t-5)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{2(s^2+2s+5)}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = \frac{1}{4}e^{-t}\sin(2t).$

Thus, $\boxed{y(t) = u_5(t) \cdot \frac{1}{4} e^{-(t-5)} \sin(2(t-5))}$. For graphing purposes we rewrite the solution as follows:

$$y(t) = \begin{cases} 0 & \text{if } 0 \le t < 5, \\ \frac{1}{4} e^{-(t-5)} \cdot \sin(2(t-5)) & \text{if } t \ge 5. \end{cases}$$

The graph is:



Response to a unit impulse at $t=5$ to a mass-spring system with $m=2$, $\gamma=4$, and $k=10$.

Ex 2 (Similar to #9, p. 343) Solve the IVP

$$y'' + y = \delta(t - \pi/2) + \delta(t - 2\pi), \quad y(0) = 1, \; y'(0) = 0,$$

and sketch the graph of the solution on $0 \le t \le 4\pi$.

Solution: We use the Laplace transform method because of the presence of Dirac deltas in the forcing term. We take the Laplace transform of the DE :

$$\mathcal{L}\{y'' + y\}(s) = \mathcal{L}\{\delta(t-\pi/2) + \delta(t-2\pi)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - s\overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} + \mathcal{L}\{y\}(s) = e^{-\frac{\pi}{2}s} + e^{-2\pi s}$$

$$(s^2+1)\mathcal{L}\{y\}(s) = s + e^{-\frac{\pi}{2}s} + e^{-2\pi s}$$

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi}{2}s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} .$$

Taking the inverse Laplace transform yields

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}$$

$$= \cos(t) + u_{\frac{\pi}{2}}(t)f(t-\tfrac{\pi}{2}) + u_{2\pi}(t)f(t-2\pi)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$. Therefore,

$$y(t) = \cos(t) + u_{\frac{\pi}{2}}(t)\sin(t-\tfrac{\pi}{2}) + u_{2\pi}(t)\sin(t-2\pi) .$$

But $\sin(t-\pi/2) = \sin(t)\cos(\pi/2) - \cos(t)\sin(\pi/2) = -\cos(t)$ and $\sin(t-2\pi) = \sin(t)$

so

$$\boxed{y(t) = \cos(t) - u_{\frac{\pi}{2}}(t)\cos(t) + u_{2\pi}(t)\sin(t)} .$$

To sketch the graph we rewrite the solution :

$$y(t) = \begin{cases} \cos(t) & \text{if } 0 \le t < \pi/2, \\ \cos(t) - \cos(t) & \text{if } \frac{\pi}{2} \le t < 2\pi, \\ \cos(t) - \cos(t) + \sin(t) & \text{if } 2\pi \le t < \infty, \end{cases}$$

$$= \begin{cases} \cos(t) & \text{if } 0 \le t < \pi/2, \\ 0 & \text{if } \pi/2 \le t < 2\pi, \\ \sin(t) & \text{if } 2\pi \le t < \infty. \end{cases}$$

The graph of the solution follows: