Chapter 7  Systems of First Order Linear DE's.

Sec. 7.1 Introduction
HW p. 359: #4, 7, 12, 22  Due: Fri., Nov. 4
Schaum's: ?

Systems of differential equations arise naturally in certain applications involving interconnected physical systems — for example, coupled mechanical or electrical vibrations (see text p. 356) or flows between a system of interconnected tanks.

(Do not write problem on the board. Ask them to read along in their textbooks. Draw Figure 7.1.6 on board.)

Ex 1 (p. 22, pp. 362–3): Consider the two interconnected tanks shown in Figure 7.1.6.

FIGURE 7.1.6 Two interconnected tanks (Problem 22).

Tank 1 initially contains 30 gal. of water and 25 oz. of salt, and Tank 2 initially contains 20 gal. of water and 15 oz. of salt. Water containing 1 oz./gal. of salt flows into Tank 1 at a rate of 1.5 gal./min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal./min. Water containing 3 oz./gal. of salt also flows into Tank 2 at a rate of 1 gal./min. (from the outside). The mixture drains from Tank 2 at a rate of 4 gal./min., of which some flows back into Tank 1.
at a rate of 1.5 gal./min., while the remainder leaves the system.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time $t$. Write down differential equations and initial conditions that model the flow process.

Solution: We apply the principle

$$\text{Net Rate of Change} = \text{Inflow Rate} - \text{Outflow Rate}$$

For each tank. Note that the volume of the mixture in each tank is constant.

**Tank 1:** \[ \frac{dQ_1}{dt} = \left( \frac{1.5 \text{ gal}}{\text{min}} \right) \left( \frac{1 \text{ oz}}{\text{gal}} \right) + \left( \frac{1.5 \text{ gal}}{\text{min}} \right) \left( \frac{Q_2 \text{ oz}}{20 \text{ gal}} \right) - \left( \frac{3 \text{ gal}}{\text{min}} \right) \left( \frac{Q_1 \text{ oz}}{30 \text{ gal}} \right) \]

**Tank 2:** \[ \frac{dQ_2}{dt} = \left( \frac{1 \text{ gal}}{\text{min}} \right) \left( \frac{3 \text{ oz}}{\text{gal}} \right) + \left( \frac{3 \text{ gal}}{\text{min}} \right) \left( \frac{Q_1 \text{ oz}}{30 \text{ gal}} \right) - \left( \frac{4 \text{ gal}}{\text{min}} \right) \left( \frac{Q_2 \text{ oz}}{20 \text{ gal}} \right) \]

Simplifying and joining the initial conditions to the system yields

\[
\begin{align*}
\frac{dQ_1}{dt} & = \frac{-1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2}, \quad Q_1(0) = 25, \\
\frac{dQ_2}{dt} & = \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3, \quad Q_2(0) = 15.
\end{align*}
\]

(Here $Q_1$ and $Q_2$ are in oz. and $t$ is in min.)
The system of DEs in #22, p.362, is an example of a first-order system of DEs in 2 unknown functions \( x_1(t), x_2(t) \):

\[
\begin{align*}
\begin{cases}
  x_1' = f_1(t, x_1, x_2) \\
  x_2' = f_2(t, x_1, x_2)
\end{cases}
\end{align*}
\]  

(\#)

If the functions \( f_1 \) and \( f_2 \) are affine-linear in the variables \( x_1 \) and \( x_2 \) then (\#) is called a linear system. That is, if (\#) can be expressed in the form

\[
\begin{align*}
\begin{cases}
  x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\
  x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t)
\end{cases}
\end{align*}
\]  

(\#\#)

then it is called a linear first-order system. If \( g_1(t) \) and \( g_2(t) \) are identically zero then (\#\#) is called homogeneous; otherwise, it is called nonhomogeneous.

**Ex. 2** The system of DEs in #22, p.362:

\[
\begin{align*}
  Q_1' &= \begin{pmatrix} \frac{1}{10} \end{pmatrix} Q_1 + \begin{pmatrix} \frac{3}{40} \end{pmatrix} Q_2 + \begin{pmatrix} \frac{3}{2} \end{pmatrix} g_1(t) \\
  Q_2' &= \begin{pmatrix} -\frac{1}{10} \end{pmatrix} Q_1 - \begin{pmatrix} \frac{1}{5} \end{pmatrix} Q_2 + \begin{pmatrix} 3 \end{pmatrix} g_2(t)
\end{align*}
\]

is a nonhomogeneous first-order linear system.
Solving linear systems can be reduced to solving a single higher-order differential equation.

**Example 3** (#10, p. 360) Transform the system

\[
\begin{align*}
\dot{x}_1 &= x_1 - 2x_2, \quad x_1(0) = -1, \\
\dot{x}_2 &= 3x_1 - 4x_2, \quad x_2(0) = 2,
\end{align*}
\]

into solving a single second-order DE with initial conditions and derive the solution to (\*).

**Solution:** We solve the first equation of (\*) for \( x_2 \):

\[
\dot{x}_2 = \frac{1}{2}x_1 - \frac{1}{2}\dot{x}_1.
\]

Using (\*), we substitute for \( x_2 \) everywhere it appears in the second equation of (\*):

\[
\left(\frac{1}{2}x_1 - \frac{1}{2}\dot{x}_1\right)' = 3x_1 - 4\left(\frac{1}{2}x_1 - \frac{1}{2}\dot{x}_1\right).
\]

Simplifying yields

\[
\frac{1}{2}\dot{x}_1 - \frac{1}{2}x_1'' = 3x_1 - 2x_1 + 2\dot{x}_1,
\]

or

\[
x_1'' - x_1' = -2x_1 - 4\dot{x}_1
\]

or

\[
x_1'' + 3x_1' + 2x_1 = 0. \quad (\text{Sometimes I get no further than this. I ask the class to finish the solution in that case.})
\]

A routine calculation shows that the general solution of this homogeneous,
The second-order linear equation is

\[ x_1(t) = c_1 e^{-t} + c_2 e^{-2t} \]

Applying the first initial condition in (\#) and the identity (\#) produces

\[ \begin{align*}
1 &= x_1(0) = c_1 + c_2 \\
\end{align*} \]

Also, differentiating (\#) gives

\[ x_1'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \]

so the two initial conditions in (\#) plus the relation (†) lead to

\[ \begin{align*}
x_2(0) &= \frac{1}{2} x_1(0) - \frac{1}{2} x_1'(0) \\
2 &= \frac{1}{2} (-1) - \frac{1}{2} (-c_1 - 2c_2) \\
\end{align*} \]

\[ \begin{align*}
2 &= c_1 + 2c_2 \\
\end{align*} \]

We solve the system (\#) by subtracting (\#) from (\#) to obtain \( b = c_2 \).

Substituting in (\#) then gives \( c_1 = -7 \). That is,

\[ x_1(t) = 6 e^{-2t} - 7 e^{-t} \]

Substituting in (†) yields

\[ x_2(t) = \frac{1}{2} x_1(t) - \frac{1}{2} x_1'(t) = \frac{1}{2} (6 e^{-2t} - 7 e^{-t}) - \frac{1}{2} (-12 e^{-2t} + 7 e^{-t}) \]

or

\[ x_2(t) = 9 e^{-2t} - 7 e^{-t} \].