Chapter 7 Systems of First Order Linear DE's.

Sec. 7.1 Introduction

HW p. 359: # 4, 7, 12, 22 Due: Fri., Nov. 4

Schaum's: ?

Systems of differential equations arise naturally in certain applications involving interconnected physical systems – for example, coupled mechanical or electrical vibrations (see text p. 356) or flows between a system of interconnected tanks.

(Don't write problem on the board. Ask them to read along in their textbooks. Draw Figure 7.1.6 on board.)

Ex1(#22, pp. 362-3): Consider the two interconnected tanks shown in Figure 7.1.6.

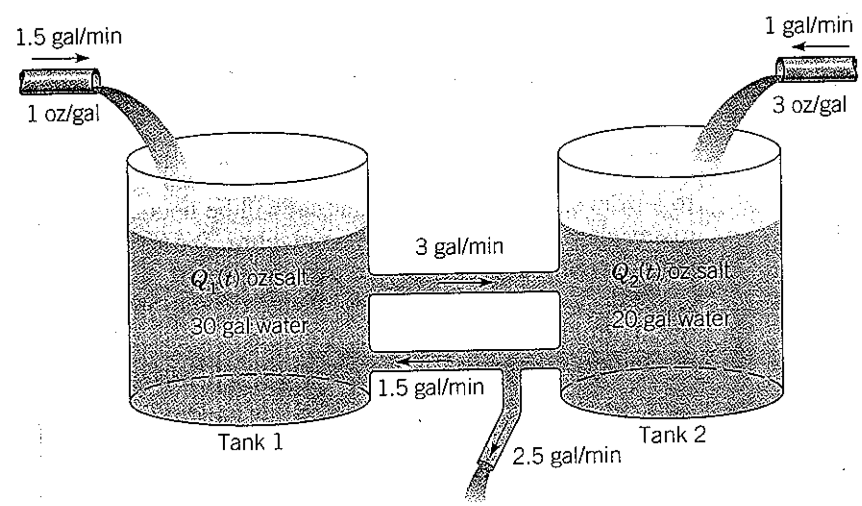

FIGURE 7.1.6 Two interconnected tanks (Problem 22).

Tank 1 initially contains 30 gal. of water and 25 oz. of salt, and Tank 2 initially contains 20 gal. of water and 15 oz. of salt. Water containing 1 oz./gal. of salt flows into Tank 1 at a rate of 1.5 gal./min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal./min. Water containing 3 oz./gal. of salt also flows into Tank 2 at a rate of 1 gal./min. (from the outside). The mixture drains from Tank 2 at a rate of 4 gal./min., of which some flows back into Tank 1 at a rate of 1.5 gal./min., while the remainder leaves the system.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process.

Solution: We apply the principle

$$\text{Net Rate of Change} = \text{Inflow Rate} - \text{Outflow Rate}$$

to the amount of salt in each tank. Note that the volume of the mixture in each tank is constant.

Tank 1 : $$\frac{dQ_1}{dt} = \left(\frac{1.5\text{ gal}}{\text{min}}\right)\left(\frac{1\text{ oz}}{\text{gal}}\right) + \left(\frac{1.5\text{ gal}}{\text{min}}\right)\left(\frac{Q_2\text{ oz}}{20\text{ gal}}\right) - \left(\frac{3\text{ gal}}{\text{min}}\right)\left(\frac{Q_1\text{ oz}}{30\text{ gal}}\right)$$

Tank 2 : $$\frac{dQ_2}{dt} = \left(\frac{1\text{ gal}}{\text{min}}\right)\left(\frac{3\text{ oz}}{\text{gal}}\right) + \left(\frac{3\text{ gal}}{\text{min}}\right)\left(\frac{Q_1\text{ oz}}{30\text{ gal}}\right) - \left(\frac{4\text{ gal}}{\text{min}}\right)\left(\frac{Q_2\text{ oz}}{20\text{ gal}}\right)$$

Simplifying and joining the initial conditions to the system yields

$$\boxed{\begin{aligned} \frac{dQ_1}{dt} &= -\frac{1}{10}Q_1 + \frac{3}{40}Q_2 + \frac{3}{2}, \qquad Q_1(0) = 25, \\ \frac{dQ_2}{dt} &= \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3, \qquad Q_2(0) = 15. \end{aligned}}$$

(Here Q_1 and Q_2 are in oz. and t is in min.)

The system of DEs in #22, p. 362 is an example of a first-order system of DEs in 2 unknown functions $x_1(t), x_2(t)$:

$$(*) \quad \begin{cases} x_1' = f_1(t, x_1, x_2) \\ x_2' = f_2(t, x_1, x_2) \end{cases}$$

If the functions f_1 and f_2 are affine-linear in the variables x_1 and x_2 then $(*)$ is called a linear system. That is, if $(*)$ can be expressed in the form

$$(**) \quad \begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{cases}$$

then it is called a <u>linear first-order system</u>. If $g_1(t)$ and $g_2(t)$ are identically zero then $(**)$ is called <u>homogeneous</u>; otherwise, it is called <u>nonhomogeneous</u>.

<u>Ex 2</u> The system of DEs in #22, p. 362:

$$Q_1' = \overset{p_{11}(t)}{\left(-\frac{1}{10}\right)}Q_1 + \overset{p_{12}(t)}{\left(\frac{3}{40}\right)}Q_2 + \overset{g_1(t)}{\left(\frac{3}{2}\right)}$$

$$Q_2' = \underset{p_{21}(t)}{\left(\frac{1}{10}\right)}Q_1 \underset{p_{22}(t)}{\left(-\frac{1}{5}\right)}Q_2 + \underset{g_2(t)}{(3)}$$

is a <u>nonhomogeneous</u> <u>first-order</u> <u>linear</u> system.

Solving first-order linear systems can be reduced to solving a single higher-order differential equation.

Ex 3 (#10, p. 360) Transform the system

$$(*)\quad \begin{cases} x_1' = x_1 - 2x_2, & x_1(0) = -1, \\ x_2' = 3x_1 - 4x_2, & x_2(0) = 2, \end{cases}$$

into solving a single second-order DE with initial conditions and derive the solution to $(*)$.

Solution: We solve the first equation of $(*)$ for x_2:

$$(\dagger)\qquad x_2 = \tfrac{1}{2}x_1 - \tfrac{1}{2}x_1'.$$

Using $(\dagger)$, we substitute for x_2 everywhere it appears in the second equation of $(*)$:

$$\left(\tfrac{1}{2}x_1 - \tfrac{1}{2}x_1'\right)' = 3x_1 - 4\left(\tfrac{1}{2}x_1 - \tfrac{1}{2}x_1'\right).$$

Simplifying yields

$$\tfrac{1}{2}x_1' - \tfrac{1}{2}x_1'' = 3x_1 - 2x_1 + 2x_1'$$

or

$$x_1'' - x_1' = -2x_1 - 4x_1'$$

or

$$x_1'' + 3x_1' + 2x_1 = 0.$$

(Sometimes I get no further than this. I ask the class to finish the solution in that case.)

A routine calculation shows that the general solution of this homogeneous, second-order linear equation is

$$(\blacksquare) \qquad x_1(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

Applying the first initial condition in $(*)$ and the identity $(\blacksquare)$ produces

$$① \qquad -1 = x_1(0) = c_1 + c_2 .$$

Also, differentiating $(\blacksquare)$ gives

$$x_1'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

so the two initial conditions in $(*)$ plus the relation $(\dagger)$ lead to

$$x_2(0) = \tfrac{1}{2}x_1(0) - \tfrac{1}{2}x_1'(0)$$

$$2 = \tfrac{1}{2}(-1) - \tfrac{1}{2}(-c_1 - 2c_2)$$

$$② \qquad 5 = c_1 + 2c_2$$

We solve the system ①-② by subtracting ① from ② to obtain $6 = c_2$.
Substituting in ① then gives $c_1 = -7$. That is,

$$\boxed{x_1(t) = 6e^{-2t} - 7e^{-t}}.$$

Substituting in $(\dagger)$ yields

$$x_2(t) = \tfrac{1}{2}x_1(t) - \tfrac{1}{2}x_1'(t) = \tfrac{1}{2}\left(6e^{-2t} - 7e^{-t}\right) - \tfrac{1}{2}\left(-12e^{-2t} + 7e^{-t}\right)$$

or

$$\boxed{x_2(t) = 9e^{-2t} - 7e^{-t}}.$$