Sec. 7.2 . Review of Matrices

HW p. 371 : # 1, 10, 21, 23 Due: Mon., Nov. 7

Schaum's : pp. 131 – 139

A matrix is a rectangular array of numbers. Examples:

Plural is "matrices"

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \end{array}\begin{array}{l} \text{row 1} \\ \text{row 2} \end{array} \quad \text{and} \quad B = \begin{array}{c} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}_{2\times 3} \\ \text{col. 1} \quad \text{col. 2} \quad \text{col. 3} \end{array}$$

Associated with each square matrix A is a number called its determinant, denoted by det A or $|A|$. Example:

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1\cdot 4 - 3\cdot 2 = -2.$$

Notation: $A = [a_{ij}]$ where a_{ij} = the element of A in row i and column j.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \Big[i(i-1)+j \Big]_{2\times 2} \qquad \left\{ \begin{aligned} a_{ij} &= i(i-1)+j \\ a_{11} &= 1(1-1)+1 = 1 \\ a_{12} &= 1(1-1)+2 = 2 \\ a_{21} &= 2(2-1)+1 = 3 \\ a_{22} &= 2(2-1)+2 = 4 \end{aligned} \right\}$$

Matrix Operations: If $A = [a_{ij}]$ then:

(transpose of A) $\quad A^T = [a_{ji}]$,

(conjugate of A) $\quad \overline{A} = [\overline{a_{ij}}]$,

(adjoint of A) $\quad A^* = \overline{A}^T$.

Ex 1 If $A = \begin{bmatrix} 3 & 1+i \\ 1-i & -2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 3 & 1-i \\ 1+i & -2 \end{bmatrix}$, $\overline{A} = \begin{bmatrix} 3 & 1-i \\ 1+i & -2 \end{bmatrix}$,

and $A^* = \overline{A}^T = \begin{bmatrix} 3 & 1+i \\ 1-i & -2 \end{bmatrix}$.

(If $A = A^T$ then A is called symmetric. If $A = A^*$ then A is called self-adjoint or hermitian.)

Addition / Subtraction of Matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}.$$

Multiplication of a Matrix by a Scalar:

$$k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}.$$

Multiplication of Two Matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where $c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j}}$.

Dot product of i^{th} row of left factor with j^{th} column of right factor.

Ex 2 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ compute

(a) $A - 2B$

(b) $3A + B$

(c) AB

(d) BA

Here students work these two at their seats.

Answers:

(a) $\begin{bmatrix} -9 & -10 \\ -11 & -12 \end{bmatrix}$ (c) $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

(b) $\begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$.

Some Properties of Matrix Algebra:

$A + B = B + A$

$A(B + C) = AB + AC$

$(AB)C = A(BC)$

Beware: $AB \neq BA$ in general.

The $n\times n$ (Multiplicative) Identity Matrix:

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

$AI_{n\times n} = I_{n\times n}A = A$ for all $n\times n$ matrices A.

(Multiplicative) Inverse Matrix: Let A be an $n\times n$ matrix. If there is an $n\times n$ matrix B such that $AB = I = BA$ then A is called invertible or nonsingular, B is called the inverse of A, and we write $B = A^{-1}$. If A does not have an inverse, then A is called singular or noninvertible.

FACTS: The square matrix A is singular if and only if $\det A = 0$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a nonsingular 2×2 matrix then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note: The inverse of a nonsingular $n\times n$ matrix can be computed by:

(a) using elementary row operations (see APP-13 & 14);

(b) using the cofactor formula $A^{-1} = \frac{1}{\det A}\left[C_{ij}\right]^T$ (see ~~APP-B~~ p.368);

(c) your calculator (see your manual).

Ex3 Compute if possible the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

Answers: $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, B is singular.

To Here
(with 40 minutes for lecture)

Differentiation and Integration of Matrix Functions:

If $A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$ then $\frac{dA}{dt} = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{bmatrix}$

and $\int_c^d A(t)\,dt = \begin{bmatrix} \int_c^d a_{11}(t)\,dt & \int_c^d a_{12}(t)\,dt \\ \int_c^d a_{21}(t)\,dt & \int_c^d a_{22}(t)\,dt \end{bmatrix}$.

Ex 4 If $A(t) = \begin{bmatrix} t & 1 \\ 3t & t^2 \end{bmatrix}$ find (a) $\frac{dA}{dt}$ and (b) $\int_0^1 A(t)\,dt$.

Answers: (a) $\begin{bmatrix} 1 & 0 \\ 3 & 2t \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 & 1 \\ 3/2 & 1/3 \end{bmatrix}$.

Matrix Differentiation Rules:

$$\frac{d}{dt}\left(CA(t)\right) = C\frac{dA}{dt}$$

$$\frac{d}{dt}\left(A(t) + B(t)\right) = \frac{dA}{dt} + \frac{dB}{dt}$$

$$\frac{d}{dt}\left(A(t)B(t)\right) = A(t)\frac{dB}{dt} + \frac{dA}{dt}B(t)$$

Omit if pressed for time.

Ex 5 Verify that $\vec{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t$ is a solution of the system $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$.