Sec. 7.7 Fundamental Matrices

HW p. 420: # 3, 6, 12, 16 Due: Mon., Nov. 29

Schaum's:

<u>Definition</u>: Consider the system $\vec{x}' = \mathbb{P}(t)\vec{x}$. (*)

If $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)$ form a fundamental set of solutions for (*) on the interval $\alpha < t < \beta$, then the matrix

$$\Psi(t) = \left[\vec{x}^{(1)}(t) \ldots \vec{x}^{(n)}(t) \right],$$

whose columns are the vectors in the fundamental set of solutions, is called a <u>fundamental matrix</u> for (*) on $\alpha < t < \beta$.

<u>Examples</u>:

① In #3 of Sec. 7.5 we found that $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$, $\vec{x}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ is a F.S.S. of $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$. Therefore a fundamental matrix for this system is

$$\Psi(t) = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & 3e^{-t} \end{bmatrix}.$$

② In Sec. 7.6 we found that

$$\vec{x}^{(1)} = e^{-t} \begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{bmatrix}, \quad \vec{x}^{(2)} = e^{-t} \begin{bmatrix} 2\cos(2t) + 2\sin(2t) \\ \sin(2t) \end{bmatrix}$$

is a F.S.S. for $\vec{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \vec{x}$. Therefore a fundamental matrix for this system is

$$\Psi(t) = \begin{bmatrix} e^{-t}(2\cos(2t) - 2\sin(2t)) & e^{-t}(2\cos(2t) + 2\sin(2t)) \\ e^{-t}\cos(2t) & e^{-t}\sin(2t) \end{bmatrix}.$$

<u>FACT 1 (p.415)</u>: If $\Psi(t)$ is a fundamental matrix for $\vec{x}' = A\vec{x}$ then Ψ is also a solution for this system; i.e. $\Psi'(t) = A\Psi(t)$ for all t.

<u>Example</u>: Consider the fundamental matrix $\Psi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$ of $\vec{x}' = \overbrace{\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}}^{A}\vec{x}$.

Then

$$\Psi'(t) - A\Psi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}' - \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & -e^{-t} \\ e^t & -3e^{-t} \end{bmatrix} - \begin{bmatrix} e^t & -e^{-t} \\ e^t & -3e^{-t} \end{bmatrix}$$

$$\stackrel{\checkmark}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is, $\Psi'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\Psi(t)$.

<u>FACT 2</u>: If $\Psi(t)$ is a fundamental matrix for $\vec{x}' = A\vec{x}$ then the general solution of the system is $\vec{x}(t) = \Psi(t)\vec{c}$ where $\vec{c}$ is an arbitrary constant vector.

<u>Reason</u>: Suppose $\Psi(t) = \left[\vec{x}^{(1)}(t) \ldots \vec{x}^{(n)}(t)\right]$ where $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)$ is F.S.S. for $\vec{x}' = A\vec{x}$. Then the general solution is ..

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + \ldots + c_n\vec{x}^{(n)}(t)$$

$$= \left[\vec{x}^{(1)}(t) \ldots \vec{x}^{(n)}(t)\right]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \Psi(t)\vec{c}$$

where $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is an arbitrary constant vector.

Ex 1 (#5, p. 420) Let $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$.

(a) Find a fundamental matrix for the system $\vec{x}' = A\vec{x}$.

(b) Find a fundamental matrix $\Phi(t)$ for this system satisfying the IVP

$$\Phi' = A\Phi, \quad \Phi(0) = I.$$

(a) $\vec{x}(t) = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda\vec{k} = A\vec{k}$. Then

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-2) + 5 = \lambda^2 + 1 \Rightarrow \lambda = \pm i.$$

Eigenvalues	Eigenvectors
$\lambda_1 = i$	$\vec{k}^{(1)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$\lambda_2 = \overline{\lambda_1}$	$\vec{k}^{(2)} = \overline{\vec{k}^{(1)}}$

Eigenvector corresponding to $\lambda = i$:

$$(A - \lambda I)\vec{k} = \vec{0} \Leftrightarrow \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \cancel{(2-i)k_1 - 5k_2 = 0} \leftarrow \text{Redundant: Eq. 1 is } 2-i \text{ times Eq. 2.} \\ k_1 - (2+i)k_2 = 0 \end{cases}$$

$\therefore\ k_1 = (2+i)k_2$ so $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} (2+i)k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$. Take $k_2 = 1$.

Then a complex solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}^{(1)}(t) = \vec{k}^{(1)}e^{\lambda_1 t} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}e^{it} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\left(\cos(t) + i\sin(t)\right).$$

A real F.S.S. of $\vec{x}' = A\vec{x}$ is

$$\tilde{\vec{x}}^{(1)}(t) = \mathrm{Re}\left(\vec{x}^{(1)}(t)\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t) = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}$$

$$\tilde{\vec{x}}^{(2)}(t) = \mathrm{Im}\left(\vec{x}^{(1)}(t)\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin(t) = \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{bmatrix}.$$

$$\therefore\ \Psi(t) = \left[\tilde{\vec{x}}^{(1)}(t),\ \tilde{\vec{x}}^{(2)}(t)\right] = \begin{bmatrix} 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix}$$ is a

fundamental matrix for $\vec{x}' = A\vec{x}$.

(b) The general solution of $\Phi' = A\Phi$ is $\Phi(t) = \Psi(t)C$ where C is an arbitrary 2x2 constant matrix. We want

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = \Phi(0) = \Psi(0)C$$

so $$C = \Psi^{-1}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

Thus $$\Phi(t) = \Psi(t)C = \begin{bmatrix} 2\cos(t)-\sin(t) & \cos(t)+2\sin(t) \\ \cos(t) & \sin(t) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t)+2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{bmatrix}.$$

I usually don't have time for this material now. I usually cover it during the special topics portion of the course.

Observe that the scalar IVP

$$x' = ax, \quad x(0) = x_0,$$

where a is a constant, has solution $x(t) = x_0 e^{at}$. This suggests that the system

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \vec{x}^0,$$

with constant coefficient matrix A, has solution

$$\vec{x}(t) = e^{At}\vec{x}^0$$

if we define the matrix exponential "correctly". Recall that if a is any real number then

$$\exp(a) = e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \cdots + \frac{a^n}{n!} + \cdots$$

<u>Definition</u>: If A is an $n\times n$ matrix then we define the exponential of A by the equation

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots \tag{22}$$

<u>Note</u>: It can be shown that the series of $n\times n$ matrices in the right member of equation (22) converges (absolutely in the Hilbert-Schmidt norm) to an $n\times n$ matrix. We call the sum of this series of matrices the matrix exponential of A and denote it by $\exp(A)$ or e^A.

<u>Ex 2</u> Compute $\exp(Bt)$ where t is a real number and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

<u>Solution</u>: $\exp(Bt) = I + \sum_{n=1}^{\infty} \frac{(Bt)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}^n}{n!}$. Note that

$\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}^2 = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} = \begin{bmatrix} t^2 & 0 \\ 0 & t^2 \end{bmatrix}$, $\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}^3 = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}^2 \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} =$

$\begin{bmatrix} t^2 & 0 \\ 0 & t^2 \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} = \begin{bmatrix} t^3 & 0 \\ 0 & -t^3 \end{bmatrix}$ and in general $\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}^n = \begin{bmatrix} t^n & 0 \\ 0 & (-1)^n t^n \end{bmatrix}$.

$$\therefore \exp(Bt) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{n=1}^{\infty} \frac{1}{n!}\begin{bmatrix} t^n & 0 \\ 0 & (-1)^n t^n \end{bmatrix} = \begin{bmatrix} 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} & 0 \\ 0 & 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \end{bmatrix}.$$

But $e^a = 1 + \sum_{n=1}^{\infty} \frac{a^n}{n!}$ so $1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t$ and $1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} = e^{-t}$.

That is, $\boxed{\exp(Bt) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.}$

FACT 3 (see pp. 416–417). If A is an $n\times n$ constant matrix then the solution to the IVP $\vec{x}'=A\vec{x}$, $\vec{x}(0)=\vec{x}^{\circ}$ is $\vec{x}(t)=\exp(At)\vec{x}^{\circ}$.

Reason: If $\vec{x}(t)=\exp(At)\vec{x}^{\circ}$ then

$$\vec{x}'(t) = \left(\exp(At)\vec{x}^{\circ}\right)' = \left(I+\sum_{n=1}^{\infty}\frac{A^n t^n}{n!}\right)'\vec{x}^{\circ} = \left(\sum_{n=1}^{\infty}\frac{\cancel{n}A^n t^{n-1}}{\cancel{n!}\,(n-1)!}\right)\vec{x}^{\circ}$$

$$= A\left(\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}\right)\vec{x}^{\circ} = A\exp(At)\vec{x}^{\circ} = A\vec{x}(t).$$ Also $\vec{x}(0)=\exp(A0)\vec{x}^{\circ}$ $= I\vec{x}^{\circ} = \vec{x}^{\circ}$. Therefore $\vec{x}(t)=\exp(At)\vec{x}^{\circ}$ is the unique (cf. Theorem 7.1.2) solution to the IVP.

In order to make effective use of FACT 3, we must be able to compute $\exp(At)$ quickly and accurately. This can be done by "diagonalization" of A when the $n\times n$ matrix A has a full set of n linearly independent eigenvectors $\vec{k}^{(1)},\dots,\vec{k}^{(n)}$. For setting $X=\left[\vec{k}^{(1)}\dots\vec{k}^{(n)}\right]$ in this case, X^{-1} exists and it can be shown (see pp. 417–418) that

$$(*)\qquad X^{-1}AX = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n\end{bmatrix},$$

a diagonal matrix. As we saw in Ex 2 it is then easy to compute the exponential of the diagonal matrix. In this event, the following result is useful.

FACT 4 (see problem #16, p.421). If D is a diagonal matrix with diagonal elements $d_1, d_2,\dots,d_n$ then $\exp(Dt)$ is also a diagonal matrix with diagonal elements $\exp(d_1 t), \exp(d_2 t), \ldots, \exp(d_n t)$.

Reason: For concreteness, we do the case when $n=2$. The case for a general $n \geq 2$ is completely analogous. Let $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ and let t be any real (or complex!) number. Then

$$(Dt)^2 = \begin{bmatrix} d_1 t & 0 \\ 0 & d_1 t \end{bmatrix}\begin{bmatrix} d_1 t & 0 \\ 0 & d_2 t \end{bmatrix} = \begin{bmatrix} d_1^2 t^2 & 0 \\ 0 & d_2^2 t^2 \end{bmatrix} \text{ and in general}$$

$$(Dt)^k = \begin{bmatrix} d_1^k t^k & 0 \\ 0 & d_2^k t^k \end{bmatrix} \text{ for } k \geq 2.$$

Consequently,

$$\exp(Dt) = I + \sum_{k=1}^{\infty} \frac{(Dt)^k}{k!} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!}\begin{bmatrix} d_1^k t^k & 0 \\ 0 & d_2^k t^k \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \sum_{k=1}^{\infty} \frac{d_1^k t^k}{k!} & 0 \\ 0 & 1 + \sum_{k=1}^{\infty} \frac{d_2^k t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{bmatrix}.$$

Ex 3| Consider the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. Use (*) (on the previous page of these notes) to show that A can be diagonalized.

Solution: In Sec. 7.5 (#3, p. 398) we showed that $\lambda_1 = 1$, $\mathbf{k}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = -1$, $\mathbf{k}^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are the eigenvalues and eigenvectors, respectively, of A. We form

$$X = \begin{bmatrix} \mathbf{k}^{(1)} & \mathbf{k}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

and compute $\Xi^{-1} = \frac{1}{\det(\Xi)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$.

Then

$$\Xi^{-1} A \Xi = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Thus A can be diagonalized by a similarity transformation

$\Xi^{-1} A \Xi$ as in (*).

<u>Ex 4</u> Use FACT 3 to solve the initial value problem,

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

<u>Solution</u>: Let $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. By FACT 3, the solution to the IVP is $\vec{x}(t) = \exp(At)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore, using the notation of <u>Ex 3</u> we have

(†) $$\Xi^{-1} \exp(At) \Xi = \Xi^{-1}\left(I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n t^n\right)\Xi = I + \sum_{n=1}^{\infty} \frac{1}{n!} \Xi^{-1} A^n \Xi t^n.$$

By <u>Ex 3</u> we have

(††) $$\Xi^{-1} A^n \Xi = \overbrace{(\Xi^{-1} A \Xi)(\Xi^{-1} A \Xi) \ldots (\Xi^{-1} A \Xi)}^{n \text{ factors}} = B^n \quad \text{where } B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore FACT 4, (†), and (††) imply

$$\Xi^{-1} \exp(At) \Xi = I + \sum_{n=1}^{\infty} \frac{1}{n!} B^n t^n = \exp(Bt) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Consequently

$$\exp(At) = \mathbb{X}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}\mathbb{X}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{t} \end{bmatrix}$$

by a routine calculation. The solution of the IVP is

$$\vec{x}(t) = \exp(At)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{t} - e^{-t} \\ 2e^{t} - 3e^{-t} \end{bmatrix},$$

or equivalently,

$$\vec{x}(t) = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-t}.$$