Mathematics 204

Fall 2010

Exam I

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. Exam I consists of this cover page and 5 pages of problems containing 7 numbered problems.
4. Once the exam begins, you will have 60 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [15] at the beginning of a problem indicates the point value of that problem is 15. The maximum possible score on this exam is 100.

	1	2	3	4	5	6	7	**Sum**
points earned								
maximum points	15	15	15	12	13	15	15	100

1.[15] State the order of each of the following differential equations. Are they linear or nonlinear? For each nonlinear equation, **CIRCLE** a term that makes it nonlinear. For each linear equation, tell whether it is homogeneous or nonhomogeneous.

Differential Equation	Order?	Linear?	Homogeneous?
$(1+y)y'' + ty' + y = e^t$	2	No	
$x' - t\ln(t)x = e^{-t}$	1	Yes	No
$x' - t\ln(x)x = e^{-t}$	1	No	
$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = 0$	2	No	
$y''' + ty' + \cos^2(t)y = 0$	3	Yes	Yes

DO NOT SOLVE ANY OF THESE EQUATIONS.

2.[15] Find the solution of the initial value problem $y' - \sin(t)\, y^2 = 0, \quad y(0) = 1/3$.

1st order, nonlinear. Use separation of variables.

$$\frac{dy}{dt} = \sin(t)\, y^2$$

$$\frac{dy}{y^2} = \sin(t)\, dt$$

$$-\frac{1}{y} + c_1 = \int \frac{dy}{y^2} = -\cos(t) + c_2$$

$$-\frac{1}{y} = -\cos(t) + c_3$$

$$\frac{-1}{-\cos(t) + c_3} = y$$

$\therefore \; y(t) = \dfrac{1}{\cos(t) + c}$ is the general solution where c is an arbitrary constant.

$$\frac{1}{3} = y(0) = \frac{1}{\cos(0) + c} = \frac{1}{1 + c}$$

so $c = 2$. Therefore

$$\boxed{y(t) = \frac{1}{2 + \cos(t)}}$$

solves the I.V.P.

3.[15] Find the general solution of the differential equation $(20+t)\, y' + 2y = \frac{3}{2}(20+t)$.

1st order, linear, nonhomogeneous. Placing the DE in standard form yields

$$(*) \quad y' + \frac{2}{20+t}\, y = \frac{3}{2}.$$

An integrating factor is

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{20+t}\,dt}$$

$$= e^{2\ln(20+t) + \cancel{c}^{\,0}}$$

$$= e^{\ln(20+t)^2}$$

$$= (20+t)^2.$$

Multiplying $(*)$ by the integrating factor produces

$$\underbrace{(20+t)^2 y' + 2(20+t) y}_{\text{Exact expression; derivative of } (20+t)^2 y} = \frac{3}{2}(20+t)^2.$$

Exact expression; derivative of $(20+t)^2 y$.

Thus

$$\frac{d}{dt}\left\{(20+t)^2 y\right\} = \frac{3}{2}(20+t)^2.$$

Integrating both sides with respect to t leads to

$$(20+t)^2 y = \int \frac{3}{2}(20+t)^2\, dt$$

$$= \frac{1}{2}(20+t)^3 + c.$$

$$\therefore \; \boxed{y(t) = \frac{1}{2}(20+t) + \frac{c}{(20+t)^2}}$$

is the general solution of the DE where c is an arbitrary constan

4.[12] Consider, **BUT DO NOT SOLVE,** the differential equation $y' = (4-y^2)y^2$.

(a) Determine the equilibrium solutions (critical points) of the differential equation.

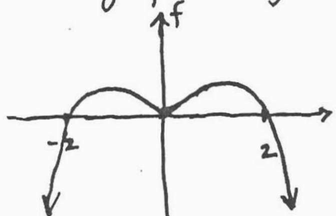
(b) Classify each equilibrium solution as asymptotically stable, unstable, or semistable. Justify your answers.

(c) If $y = y(t)$ denotes the solution of the initial value problem $y' = (4-y^2)y^2$, $y(0) = 1$, use your answers in part (b) to determine $\lim_{t\to\infty} y(t)$.

(a) $0 = (4-y^2)y^2 = (2-y)(2+y)y^2$

implies $\boxed{y = 2,\ y = -2, \text{ and } y = 0}$ are the equilibrium solutions.

(b) The graph of $f(y) = (4-y^2)y^2$ is:



and the phase line of the DE is:

Therefore $\boxed{y = 2 \text{ is asymptotically stable}}$

$\boxed{y = 0 \text{ is asymptotically semistable}}$;

$\boxed{y = -2 \text{ is asymptotically unstable}}$.

(c) Since $0 < y(0) < 2$ and $y = 2$ is an asymptotically stable equilibrium solution,

$\boxed{\lim_{t\to\infty} y(t) = 2}$.

5.[13] A 100 gallon tank originally contains 20 gallons of water and 5 pounds of salt. Then water containing $1/2$ pound of salt per gallon is poured into the tank at a rate of 3 gallons per minute, and the well-stirred mixture leaves at a rate of 2 gallons per minute. Set up, **BUT DO NOT SOLVE,** an initial value problem that models the amount of salt in the tank for times t between 0 and 80 minutes.

Let $A(t)$ denote the number of pounds of salt in the tank at time t minutes.

We use:

Net Rate = Rate In − Rate Out.

Therefore

$$\frac{dA}{dt} = \left(\frac{\frac{1}{2}\text{ lb.}}{\text{gal.}}\right)\left(\frac{3\text{ gal.}}{\text{min.}}\right) - \left(\frac{A(t)\text{ lb.}}{V(t)\text{ gal.}}\right)\left(\frac{2\text{ gal.}}{\text{min.}}\right)$$

where $V(t)$ denotes the volume of solution in the tank at time t. Since the tank originally contains 20 gallons of solution and the tank gains 1 gallon of solution in each minute, $V(t) = 20 + t$. Therefore the I.V.P. that models the amount of salt in the tank for $0 \le t \le 80$ is

$$\boxed{\frac{dA}{dt} = \frac{3}{2} - \left(\frac{2}{20+t}\right)A, \quad A(0) = 5.}$$

(cf. #3 for the general solution of the DE.)

6.[15] (a) Find the general solution of $2y'' - 5y' - 3y = 0$.

(b) Find the general solution of $y'' + 4y' + 5y = 0$.

(c) Do the functions $f(t) = t$ and $g(t) = te^{t}$ form a fundamental set of solutions of $t^2y'' - t(t+2)y' + (t+2)y = 0$ on the interval $t > 0$? Give reasons for your answer.

(a) $y = e^{rt}$ leads to $2r^2 - 5r - 3 = 0$ or $(2r+1)(r-3) = 0$ so $r = -\frac{1}{2}$ or $r = 3$. Gen. soln.: $\boxed{y = c_1 e^{-t/2} + c_2 e^{3t}}$ where c_1 and c_2 are arbitrary constants.

(b) $y = e^{rt}$ leads to $r^2 + 4r + 5 = 0$ so $r = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$. The general solution is $y = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$ where $\lambda = -2$ and $\mu = 1$. Therefore $\boxed{y - e^{2t}(c_1\cos(t) + c_2\sin(t))}$ is the general solution where c_1 and c_2 are arbitrary constants.

5 pts. to here

(c) Note that the number of functions $= 2 =$ the order of the differential equation

$$t^2 f''(t) - t(t+2)f'(t) + (t+2)f(t) = 0 - t(t+2)\cdot 1 + (t+2)t \overset{\checkmark}{=} 0$$

so $f(t) = t$ solves the differential equation for $t > 0$.

$$t^2 g''(t) - t(t+2)g'(t) + (t+2)g(t) = t^2(t+2)e^t - t(t+2)(t+1)e^t + (t+2)te^t$$
$$= (t+2)e^t\left[t^2 - t(t+1) + t\right]$$
$$\overset{\checkmark}{=} 0$$

so $g(t) = te^t$ solves the differential equation for $t > 0$.

$$W(f,g)(t) = \begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix} = t(t+1)e^t - te^t = t^2e^t \neq 0 \text{ for } t > 0.$$

Therefore, $\boxed{\text{yes,}}$ the functions $f(t) = t$, $g(t) = te^t$ form a fundamental set of solutions of $t^2y'' - t(t+2)y' + (t+2)y = 0$ on the interval $t > 0$.

7.[15] Given that $y_1(x) = e^x$ is a solution to $xy'' + (1-2x)y' + (x-1)y = 0$ on the interval $x > 0$, use reduction of order to find a second solution that is not a constant multiple of y_1.

Assume that a second solution has the form $y_2(x) = u(x)y_1(x) = e^x u(x)$ where $u = u(x)$ is an appropriately chosen nonconstant function of x. Then

$y_2' = e^x u + e^x u'$ and $y_2'' = e^x u + e^x u' + e^x u' + e^x u'' = e^x(u + 2u' + u'')$.

We want $xy_2'' + (1-2x)y_2' + (x-1)y_2 = 0$ so substituting from above yields

$$xe^x(u + 2u' + u'') + (1-2x)e^x(u+u') + (x-1)e^x u = 0$$

or

$$x(u + 2u' + u'') + (1-2x)(u+u') + (x-1)u = 0,$$

so

$$xu'' + \underbrace{(2x + 1 - 2x)}_{1}u' + \underbrace{(x + 1 - 2x + x - 1)}_{0}u = 0.$$

Therefore $xu'' + u' = 0$. If we set $v = u'$ then $v' = u''$ so the DE in u is equivalent to the first order, linear DE in v: $xv' + v = 0$. The left member of this DE is exact; it is the derivative of xv. Therefore

$\frac{d}{dx}(xv) = 0$, so integrating yields $xv = c_1$. But $v = u'$ so substituting yields $xu' = c_1$ and hence $u = \int \frac{c_1}{x}\,dx = c_1 \ln(x) + c_2$.

Therefore $y_2(x) = u(x)e^x = (c_1 \ln(x) + c_2)e^x = c_1 e^x \ln(x) + c_2 e^x$. If we take $c_1 = 1$ and $c_2 = 0$ then we get a "clean" second solution to the DE that is not a constant multiple of $y_1(x) = e^x$ on the interval $x > 0$:

$$\boxed{y_2(x) = e^x \ln(x)}.$$

2010 Fall Semester, Math 204 Hour Exam I, Master List

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100 F's

✓ A Wintz (41)
✓ B Wintz (35)
✓ C Wintz (39)
✓ D Willinger (42)
✓ E Heim (44)
✓ F Grow (37)
✓ G Fitch (42)
✓ H Fitch (35)
✓ J He (35)
✓ K Heim (37)
✓ L Singler (39)

Number taking exam: 426
Median: 77
Mean: 72.79
Standard Deviation: 19.46

Number receiving A's: 102 — 23.9%
Number receiving B's: 79 — 18.5%
Number receiving C's: 74 — 17.4%
Number receiving D's: 71 — 16.7%
Number receiving F's: 100 — 23.5%