**Mathematics 204**

**Spring 2010**

**Exam I**

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. Exam I consists of this cover page and 6 pages of problems containing 7 numbered problems.
4. Once the exam begins, you will have 60 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Sum |
|---|---|---|---|---|---|---|---|---|---|
| points earned | | | | | | | | | |
| maximum points | 2 | 13 | 12 | 16 | 17 | 16 | 8 | 16 | 100 |

1.[13] Determine the order of each differential equation and state whether it is linear or nonlinear. For each nonlinear equation, circle a term that makes the equation nonlinear.

(a) $(1-x)y''-4xy'+5y=\cos(x)$ 2nd order, linear

(b) $\ln(x)\dfrac{d^3y}{dx^3}-\left(\dfrac{dy}{dx}\right)^4+y=0$ 3rd order, nonlinear

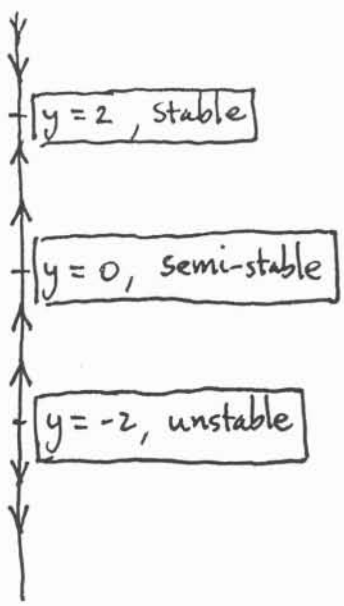
(c) $\sin(t)y''-\cos(t)y'=y$ 2nd order, linear

(d) $\dfrac{d^2y}{dx^2}=\sqrt{\dfrac{dy}{dx}}$ 2nd order, nonlinear

(e) $y'''+ty'+\cos(y)y=0$ 3rd order, nonlinear

2.[12] Find the equilibrium solutions and sketch the phase portrait of the differential equation $y'=y^2(4-y^2)$. Classify each equilibrium solution as asymptotically stable, unstable, or semi-stable.

$0=y'=y^2(4-y^2)=y^2(2-y)(2+y)$ $\boxed{\text{Equilibrium solutions: } y=0,\ y=-2,\ y=2}$




| Interval | Sign of $y'=y^2(4-y^2)$ |
|---|---|
| $(2,\infty)$ | $-$ |
| $(0,2)$ | $+$ |
| $(-2,0)$ | $+$ |
| $(-\infty,-2)$ | $-$ |

Phase Portrait

3.[16] Solve the differential equation $y' = \dfrac{1+xy}{x^2}$ for $x > 0$.

1 – to here $\quad x^2 y' - xy = 1$ (Linear, first-order)

2 pts to here $\quad y' - \frac{1}{x} y = \frac{1}{x^2}$

6 pts to here (1+2+[illegible]) $\quad \mu = e^{\int p(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln(x) + \cancel{C}^{\,0}} = e^{\ln(x^{-1})} = x^{-1}.$

7 pts to here $\quad x^{-1}\left(y' - \frac{1}{x} y\right) = x^{-1} \cdot \frac{1}{x^2}$

$$\frac{d}{dx}(x^{-1}y) = \underbrace{x^{-1}y' - x^{-2}y}_{\text{Exact}} = x^{-3}$$

[illegible] pts to here $\quad \int \frac{d}{dx}(x^{-1}y)\,dx = \int x^{-3}\,dx$

15 pts to here ([illegible]) $\quad x^{-1}y = \dfrac{x^{-2}}{-2} + C$

$$y = x\left(-\frac{1}{2x^2} + C\right)$$

–3 if they have c instead of cx.

16 pts to here $\quad \boxed{y = cx - \dfrac{1}{2x}}$

If they used the formula

(*) $\quad y(x) = \dfrac{1}{\mu(x)} \int \mu(x) [illegible]\,dx$

then full credit was given if $\mu(x)$ was computed correctly and the formula (*) was implemented correctly. No credit (including the partial credit for $\mu$ in the key above) was given if the formula (*) was implemented with an incorrect $\mu(x)$.

4.[17] Find an explicit solution of the initial value problem $y' = \dfrac{2x+1}{2y}$, $y(-2) = -1$.

$2y\,dy = (2x+1)\,dx$ (Variables Separable)

$\int 2y\,dy = \int (2x+1)\,dx$

$y^2 = x^2 + x + c$

$y = \pm\sqrt{x^2 + x + c}$

In order to satisfy the initial condition $y(-2) = -1$, we must select the negative root.

$-1 = y(-2) = -\sqrt{(-2)^2 + (-2) + c}$

$-1 = -\sqrt{2 + c}$

$c = -1$

$\therefore \boxed{y(x) = -\sqrt{x^2 + x - 1}}$

5.[16] (a) When interest is compounded continuously, the amount $A(t)$ of money in a savings account at time $t$ increases at a rate proportional to the amount currently in the account. Write a differential equation that models the amount of money in such a savings account.

$$\boxed{\frac{dA}{dt} = kA} \quad (k \text{ is a proportionality constant})$$

(b) The constant of proportionality in the model in part (a) is called the annual interest rate. If \$5000 is initially deposited in a savings account in which interest is compounded continuously at an annual interest rate of 5%, find the time needed for the initial deposit to double.

$$A(0) = 5000$$

$$\frac{dA}{dt} = kA \Rightarrow \int \frac{dA}{A} = \int k\,dt \Rightarrow \ln A = kt + c$$

$$\Rightarrow A(t) = e^{kt+c} = Be^{kt} \quad (B = e^{c} \text{ is a constant.})$$

$$5000 = A(0) = Be^{0} = B. \quad \text{Also } k = 0.05 = 1/20 \text{ so}$$

$$A(t) = 5000e^{t/20}.$$

We need to find the time $t$ when $A(t) = 2 \cdot 5000$. I.e. we must solve

$$10{,}000 = A(t) = 5000e^{t/20},$$

$$\Rightarrow \quad 2 = e^{t/20},$$

$$\Rightarrow \quad \ln(2) = \frac{t}{20},$$

$$\Rightarrow \quad \boxed{t = 20\ln(2) \doteq 13.86 \text{ years}}.$$

6.[8] Do not attempt to solve the following differential equations on the given intervals. Instead, consider the facts known about each and state whether or not $y_1$ and $y_2$ are guaranteed to form a fundamental set of solutions in each case. Give reasons for your answers.

(a) $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y''' + 7y'' - 11y' + xy = 0$ on $(0, \infty)$.

No, they are not guaranteed to form a F.S.S. for this DE on $(0,\infty)$ because the DE is third order and so <u>three</u> linearly independent solutions of the DE on $(0,\infty)$ are required for a F.S.S.

(b) $y_1(x)$ and $y_2(x)$ are solutions of $(x-1)y'' + xy' + 5y = 0$ on $(1, \infty)$.

No, because $y_1$ and $y_2$ may not form a <u>linearly</u> <u>independent</u> set of two solutions to the second-order DE on $(1,\infty)$.

(c) $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $x^2y'' - xy' + y = 0$ on $(0, \infty)$.

Yes, $y_1$ and $y_2$ are guaranteed to form a F.S.S. to the second-order DE on $(0,\infty)$. They are linearly independent on $(0,\infty)$, they are solutions to the DE on $(0,\infty)$, and there are two solutions, matching the order of the DE.

(d) $y_1(x)$ and $y_2(x)$ are linearly independent functions on $(0, \infty)$; $x^2y'' + (x-1)y' + xy = 0$ on $(0, \infty)$.

No, $y_1$ and $y_2$ are not guaranteed to form a F.S.S. of the DE on $(0,\infty)$ because one or both may not actually be <u>solutions</u> to the DE on $(0,\infty)$.

7.[16] One solution of the differential equation $x^2y''+5xy'+4y=0$ is $y_1(x)=x^{-2}$. Use reduction of order to find a second linearly independent solution $y_2(x)$ of this differential equation for $x>0$. Verify that $y_1(x)$ and $y_2(x)$ are linearly independent on $(0,\infty)$.

Assume $y_2(x)=u(x)y_1(x)$ is a solution to the DE on $(0,\infty)$ where $u$ is a nonconstant function. Then $y_2(x)=x^{-2}u(x)$,

$$y_2'(x)=-2x^{-3}u(x)+x^{-2}u'(x),$$

and $y_2''(x)=6x^{-4}u(x)-2x^{-3}u'(x)-2x^{-3}u'(x)+x^{-2}u''(x)=6x^{-4}u(x)-4x^{-3}u'(x)+x^{-2}u''(x)$.

We want $y_2$ to solve the DE so

$$0=x^2y_2''+5xy_2'+4y_2$$

$$0=x^2\left(6x^{-4}u(x)-4x^{-3}u'(x)+x^{-2}u''(x)\right)+5x\left(-2x^{-3}u(x)+x^{-2}u'(x)\right)+4x^{-2}u(x)$$

$$0=\underbrace{(6x^{-2}-10x^{-2}+4x^{-2})}_{0}u(x)+(\underbrace{-4x^{-1}+5x^{-1}}_{x^{-1}})u'(x)+u''(x)$$

$$0=x^{-1}u'(x)+u''(x)$$

$$0=v'(x)+x^{-1}v(x)\qquad (\text{where } v(x)=u'(x))$$

First-order linear; an integrating factor is $\mu(x)=e^{\int p(x)dx}=e^{\int x^{-1}dx}=e^{\ln(x)+\cancel{c}^{\,0}}=x$.

$$\therefore\quad x\cdot 0=x\left(v'(x)+x^{-1}v(x)\right)=\underbrace{xv'(x)+v(x)}_{\text{Exact}}=(xv(x))'$$

$$\int (xv(x))'dx=\int 0\,dx$$

$$xv(x)=c_1$$

$$(u'(x)=)\ v(x)=\frac{c_1}{x}$$

$$\Rightarrow u(x)=\int\frac{c_1}{x}dx=c_1\ln(x)+c_2$$

$\therefore\ y_2(x)=x^{-2}u(x)=x^{-2}(c_1\ln(x)+c_2)=c_1x^{-2}\ln(x)+\overset{y_1(x)}{c_2x^{-2}}$. Therefore we may take $c_1=1, c_2=0$ to get a second l.i. solution to the DE: $\boxed{y_2(x)=x^{-2}\ln(x)}$. Verification:

$$W(y_1,y_2)(x)=\begin{vmatrix} x^{-2} & x^{-2}\ln(x) \\ -2x^{-3} & -2x^{-3}\ln(x)+x^{-2}\cdot x^{-1}\end{vmatrix}=\cancel{-2x^{-5}\ln(x)}+x^{-5}+\cancel{2x^{-5}\ln(x)}=x^{-5}\neq 0 \text{ on } (0,\infty).$$