Mathematics 204

Fall 2011

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transforms.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work.** **No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals and determinant computations must be done by hand.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

| | 1 | 2 | 3 | 4 | 5 | **Sum** |
|---|---|---|---|---|---|---|
| **points earned** | | | | | | |
| **maximum points** | 20 | 20 | 20 | 20 | 20 | 100 |

1.[20] Find the general solution of $y^{(4)} - 3y'' - 4y = e^{-t}$ on the interval $-\infty < t < \infty$.

$y = e^{rt}$ in $y^{(4)} - 3y'' - 4y = 0$ leads to $r^4 - 3r^2 - 4 = 0 \Rightarrow (r^2-4)(r^2+1) = 0$
$\Rightarrow r = \pm 2,\ r = \pm i$. Therefore $y_c(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(t) + c_4\sin(t)$ is the general solution of the associated homogeneous equation.

Since $\alpha = -1$ is not a root of the characteristic equation, a trial form for a particular solution to the nonhomogeneous equation is $y_p(t) = Ae^{-t}$. [We are using the method of undetermined coefficients here.] Then $y_p' = -Ae^{-t}$, $y_p'' = Ae^{-t}$, $y_p''' = -Ae^{-t}$, $y_p^{(4)} = Ae^{-t}$.
We want to choose A so that $y_p^{(4)} - 3y_p'' - 4y_p = e^{-t}$. Substituting the expressions for $y_p$, $y_p''$, and $y_p^{(4)}$ we have:

$$Ae^{-t} - 3Ae^{-t} - 4Ae^{-t} = e^{-t}$$

$$\Rightarrow \quad -6Ae^{-t} = e^{-t}$$

$$\Rightarrow \quad A = -\tfrac{1}{6}.$$

Therefore $y_p(t) = -\frac{1}{6}e^{-t}$ is a particular solution of the nonhomogeneous equation.
The general solution of the nonhomogeneous equation on $-\infty < t < \infty$ is

$$y(t) = y_c(t) + y_p(t)$$

or

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3\cos(t) + c_4\sin(t) - \tfrac{1}{6}e^{-t}}$$

where $c_1, c_2, c_3$, and $c_4$ are arbitrary constants.

2.[20] Solve the initial value problem $t^2y'' - 2ty' + 2y = 3t^2$, $y(1) = 0$, $y'(1) = 4$.

This is an Euler equation. Then $y(t) = t^m$ in the associated homogeneous equation $t^2y'' - 2ty' + 2y = 0$ leads to $m(m-1) - 2m + 2 = 0$ or $m^2 - 3m + 2 = 0$ or $(m-1)(m-2) = 0$ so $m = 1$ or $m = 2$. Consequently $y_c(t) = c_1 t + c_2 t^2$ is the general solution of the associated homogeneous equation. To find a particular solution of the nonhomogeneous equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 3$$

we use variation of parameters: $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where $y_1(t) = t$, $y_2(t) = t^2$, $u_1(t) = \int \frac{-g(t)y_2(t)}{W(y_1,y_2)(t)}\,dt$, and $u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1,y_2)(t)}\,dt$.

Here $g(t) = 3$ and $W(y_1,y_2)(t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$. Therefore

$$u_1(t) = \int \frac{-3t^2}{t^2}\,dt = -3t + \cancelto{0}{c_1} \quad \text{and} \quad u_2(t) = \int \frac{3t}{t^2}\,dt = 3\ln(t) + \cancelto{0}{c_2}.$$

Hence $y_p(t) = (-3t)(t) + (3\ln(t))(t^2) = -3t^2 + 3t^2\ln(t)$. The general solution of the nonhomogeneous equation is $y = y_c + y_p = c_1 t + c_2 t^2 - 3t^2 + 3t^2\ln(t)$ or $y(t) = c_1 t + \tilde{c}_2 t^2 + 3t^2\ln(t)$ where $c_1$ and $\tilde{c}_2$ are arbitrary constants. We must choose the constants so the initial conditions are satisfied.

(1) $\quad 0 = y(1) = c_1 + \tilde{c}_2 + 3\ln(1) = c_1 + \tilde{c}_2$

$$y'(t) = c_1 + 2\tilde{c}_2 t + 3t\ln(t) + 3t^2 \cdot \frac{1}{t} = c_1 + 2\tilde{c}_2 t + 3t\ln(t) + 3t$$

(2) $\quad 4 = y'(1) = c_1 + 2\tilde{c}_2 + 3\ln(1) + 3 = c_1 + 2\tilde{c}_2 + 3$

Subtracting equation (1) from (2) yields $4 = \tilde{c}_2 + 3 \Rightarrow \tilde{c}_2 = 1$. Substituting this in (1) gives $c_1 = -1$. Therefore

$$\boxed{y(t) = t^2 - t + 3t^2\ln(t)}$$

solves the initial value problem on $0 < t < \infty$.

3. (Please use 9.8 meters per second per second as the acceleration of gravity in this problem.) A 20 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 50 centimeters beyond its natural length. The body is acted on by an external force of $10\cos(2t)$ Newtons and moves in a medium with a damping constant of 100 Newton seconds per meter.

(a) [15] If the body is set in motion from its equilibrium position with an upward velocity of 20 centimeters per second, **SET UP, BUT DO NOT SOLVE**, an initial value problem describing the motion of the body.

Let $u(t)$ denote the body's vertical displacement from the static equilibrium position at time $t$. (We measure $u(t)$ in meters and $t$ in seconds.) Then $mu'' + \gamma u' + ku = f(t)$ where $m = 20$ kg, $\gamma = 100$ N·s/m, $f(t) = 10\cos(2t)$ N, and the stiffness constant $k$ of the spring satisfies $ku_0 = mg$ so $k = \frac{mg}{u_0} = \frac{(20)(9.8)}{.5} = 392$ N/m. Therefore

$$\boxed{20u'' + 100u' + 392u = 10\cos(2t), \quad u(0) = 0, \quad u'(0) = -0.2}$$

is an initial value problem which models the body's motion.

(b) [5] If the given downward external force is replaced by a force of $10\cos(\omega t)$ Newtons, find the value of the frequency $\omega$ which will cause resonance or explain why there is no such frequency.

Resonance does not occur regardless of the frequency $\omega$ of the external force $f(t) = 10\cos(\omega t)$. The reason for this lack of resonance is the fact that the motion is (strongly) damped. Consequently, the general solution of the DE is

$$u(t) = u_c(t) + u_p(t) = e^{-\frac{5}{2}t}\left(c_1\cos\left(\frac{\sqrt{1335}}{10}t\right) + c_2\sin\left(\frac{\sqrt{1335}}{10}t\right)\right) + A\cos(\omega t) + B\sin(\omega t)$$

where $c_1$ and $c_2$ are arbitrary constants and A and B are "appropriately chosen" constants. Note that $|u(t)|$ is bounded as $t \to \infty$ so resonance cannot occur.

[ Calculations: $20r^2 + 100r + 392 = 0 \Rightarrow r = \frac{-100 \pm \sqrt{10{,}000 - 4(20)(392)}}{2(20)} = \frac{-5}{2} \pm \frac{i\sqrt{1335}}{10}$. ]

4.[20] Solve the initial value problem $y'' + 4y = 4u_\pi(t)$, $y(0) = 1$, $y'(0) = 0$. Write your solution as a piecewise defined function and sketch its graph on the interval $0 \le t \le 2\pi$.

Assume that $y = y(t)$ is a solution to the I.V.P. so $y''(t) + 4y(t) = 4u_\pi(t)$, $y(0) = 1$, $y'(0) = 0$.

Taking the Laplace transform gives

$$\mathcal{L}\{y'' + 4y\}(s) = \mathcal{L}\{4u_\pi\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - s\cancelto{1}{y(0)} - \cancelto{0}{y'(0)} + 4\mathcal{L}\{y\}(s) = \frac{4e^{-\pi s}}{s}.$$

Solving for $\mathcal{L}\{y\}$: $$(s^2+4)\mathcal{L}\{y\}(s) = s + \frac{4e^{-\pi s}}{s}$$

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+4} + \frac{4}{s(s^2+4)} \cdot e^{-\pi s}.$$

Then $$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)} \cdot e^{-\pi s}\right\}$$

$$= \cos(2t) + u_\pi(t) f(t-\pi)$$

$$\frac{4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$4 = A(s^2+4) + (Bs+C)s$$

$s = 0 \Rightarrow 4 = 4A \Rightarrow A = 1$

$s = 2i \Rightarrow 4 = (2Bi + C)(2i)$

$\Rightarrow C = 0,\ B = -1$

where $$f(t) = \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$$

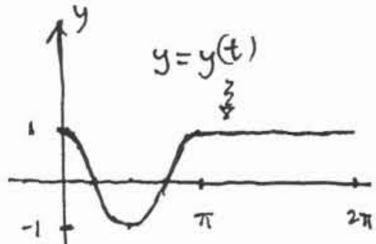
so $f(t) = 1 - \cos(2t)$.

Therefore $$\boxed{y(t) = \cos(2t) + u_\pi(t)\left[1 - \cos(2(t-\pi))\right]}.$$

To sketch the graph of $y$, we first write it as a piecewise defined function.

$$y(t) = \begin{cases} \cos(2t) & \text{if } 0 \le t < \pi, \\ \cos(\cancel{2t}) + [1 - \cos(\cancel{2t - 2\pi})] & \text{if } \pi < t < \infty, \end{cases}$$

$$= \begin{cases} \cos(2t) & \text{if } 0 \le t < \pi, \\ 1 & \text{if } \pi < t < \infty. \end{cases}$$



5.[20] Solve the initial value problem $y''+3y'+2y=\delta(t-2),\ \ y(0)=0,\ y'(0)=1.$

Suppose that $y=y(t)$ is a solution to the I.V.P. Then $y''(t)+3y'(t)+2y(t)=\delta(t-2)$, $y(0)=0$, $y'(0)=$

Taking the Laplace transform of both sides of the DE we find

$$\mathcal{L}\{y''+3y'+2y\}(s) = \mathcal{L}\{\delta(t-2)\}(s)$$

$$s^2\mathcal{L}\{y\}(s) - s\cancel{y(0)}^{0} - y'(0)^{1} + 3\left(s\mathcal{L}\{y\}(s) - y(0)^{0}\right) + 2\mathcal{L}\{y\}(s) = e^{-2s}$$

Solving for $\mathcal{L}\{y\}(s)$ we have

$$(s^2+3s+2)\mathcal{L}\{y\}(s) = 1 + e^{-2s}$$

$$\mathcal{L}\{y\}(s) = \frac{1}{s^2+3s+2} + \frac{e^{-2s}}{s^2+3s+2}.$$

Therefore $$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\cdot e^{-2s}\right\}$$

$$= f(t) + f(t-2)u_2(t)$$

$$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s+2$$

$$s=-1 \Rightarrow 1 = B(1)$$

$$s=-2 \Rightarrow 1 = A(-1)$$

where $$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-1}{s+2} + \frac{1}{s+1}\right\}$$

$$= e^{-t} - e^{-2t}.$$

Thus $$\boxed{y(t) = e^{-t} - e^{-2t} + \left[e^{-(t-2)} - e^{-2(t-2)}\right]u_2(t)}.$$

SHORT TABLE OF LAPLACE TRANSFORMS

| $f(t)$ | $\mathcal{L}\{f(t)\} = F(s)$ |
|---|---|
| 1. $e^{at}$ | $\dfrac{1}{s-a}$ |
| 2. $t^n$ | $\dfrac{n!}{s^{n+1}}, \quad n = 0,1,2,3....$ |
| 3. $\sin(bt)$ | $\dfrac{b}{s^2+b^2}$ |
| 4. $\cos(bt)$ | $\dfrac{s}{s^2+b^2}$ |
| 5. $f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - ... - f^{(n-1)}(0)$ |
| 6. $e^{ct} f(t)$ | $F(s-c)$ |
| 7. $u_c(t) f(t-c)$ | $e^{-cs} F(s)$ |

# 2011 Fall Semester, Math 204 Hour Exam 2, Master List

| Score | Tally | Score | Tally | Score | Tally |
|---|---|---|---|---|---|
| 100 | \| | 59 | 卌 \| | 19 | |
| 99 | \|\| | 58 | 卌 \|\|\| | 18 | |
| 98 | \| | 57 | 卌 \|\|\|\| | 17 | |
| 97 | \|\| | 56 | 卌 卌 \|\|\| | 16 | |
| 96 | 卌 | 55 | 卌 | 15 | |
| 95 | \|\|\|\| | 54 | 卌 \|\| | 14 | |
| 94 | 卌 \| | 53 | 卌 | 13 | |
| 93 | \|\|\|\| | 52 | \|\|\| | 12 | |
| 92 | 卌 | 51 | 卌 \|\|\|\| | 11 | |
| 91 | 卌 | 50 | \|\| | 10 | |
| 90 | 卌 | 49 | 卌 卌 | 9 | |
| 89 | \|\|\|\| | 48 | \|\|\|\| | 8 | |
| 88 | 卌 | 47 | \|\|\| | 7 | |
| 87 | 卌 | 46 | 卌 | 6 | |
| 86 | 卌 \|\|\| | 45 | \|\| | 5 | |
| 85 | 卌 \| | 44 | 卌 \| | 4 | |
| 84 | 卌 \|\|\|\| | 43 | 卌 \| | 3 | |
| 83 | \|\|\| | 42 | 卌 \| | 2 | |
| 82 | 卌 \| | 41 | 卌 | 1 | |
| 81 | 卌 卌 \|\|\|\| | 40 | 卌 \| | 0 | |
| 80 | 卌 \|\|\| | 39 | \| | | |
| 79 | \|\|\|\| | 38 | \|\|\|\| | | |
| 78 | 卌 \| | 37 | 卌 | | |
| 77 | 卌 卌 \| | 36 | \|\| | | |
| 76 | 卌 \| | 35 | 卌 | | |
| 75 | \|\|\|\| | 34 | \|\|\| | | |
| 74 | 卌 卌 卌 \|\| | 33 | \| | | |
| 73 | 卌 | 32 | \| | | |
| 72 | 卌 \|\| | 31 | 卌 | | |
| 71 | 卌 卌 | 30 | | | |
| 70 | 卌 卌 \| | 29 | \|\| | | |
| 69 | 卌 卌 | 28 | \| | | |
| 68 | 卌 \|\| | 27 | | | |
| 67 | 卌 卌 \| | 26 | | | |
| 66 | 卌 | 25 | \| | | |
| 65 | 卌 | 24 | \|\| | | |
| 64 | 卌 卌 \| | 23 | | | |
| 63 | 卌 \|\| | 22 | \| | | |
| 62 | 卌 卌 卌 | 21 | | | |
| 61 | 卌 \|\| | 20 | \| | | |
| 60 | 卌 卌 \| | | | | |

155 F's

81 C's

89 D's

Number taking exam: 433
Median: 67
Mean: 65.40
Standard Deviation: 17.77

Number receiving A's: 40 — 9.2%
Number receiving B's: 68 — 15.7
Number receiving C's: 81 — 18.7
Number receiving D's: 89 — 20.6
Number receiving F's: 155 — 35.8