Mathematics 204

Spring 2010

Exam III

[1] Your Printed Name: Dr. Grow

Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. Exam III consists of this cover page, 5 pages of problems containing 5 numbered problems, and a short table of Laplace transform formulas.
4. Once the exam begins, you will have 60 minutes to complete your solutions.
5. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show. In particular, all integrals, partial fraction decompositions, and matrix computations must be done by hand.
6. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
7. The symbol [20] at the beginning of a problem indicates the point value of that problem is 20. The maximum possible score on this exam is 100.

	0	1	2	3	4	5	**Sum**
points earned							
maximum points	1	20	20	20	19	20	100

1.[20] Solve the initial value problem $y''-4y'+5y=\delta(t-2\pi)$, $y(0)=0$, $y'(0)=0$. Please express your final answer without any unit step functions.

We use the method of Laplace transforms.

$$\mathcal{L}\{y''-4y'+5y\}(s) = \mathcal{L}\{\delta(t-2\pi)\}(s)$$

formulas 6, 7, and 8 on the short table

$$s^2\mathcal{L}\{y\}(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} - 4\left(s\mathcal{L}\{y\}(s) - \cancel{y(0)}^{0}\right) + 5\mathcal{L}\{y\}(s) \stackrel{*}{=} e^{-2\pi s}$$

$$\mathcal{L}\{y\}(s) = \frac{e^{-2\pi s}}{s^2-4s+5}$$

formula 11 on short table

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s-2)^2+1}\right\} \stackrel{*}{=} f(t-2\pi)\,\mathcal{U}(t-2\pi)$$

formula 9 on short table

where $f(t) = \mathcal{L}^{-1}\left\{\overbrace{\frac{1}{(s-2)^2+1}}^{G(s-2)}\right\} \stackrel{*}{=} e^{2t}g(t)$ with $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$
$= \sin(t)$.

Therefore $f(t) = e^{2t}\sin(t)$. Consequently,

$$y(t) = f(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$y(t) = e^{2(t-2\pi)}\sin(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$\boxed{y(t) = \begin{cases} 0 & \text{if } t < 2\pi, \\ e^{2(t-2\pi)}\sin(t) & \text{if } t \geq 2\pi. \end{cases}}$$

2.[20] Solve the integrodifferential equation $y'(t)=1-\int_0^t e^{-2\tau}y(t-\tau)\,d\tau$ subject to the initial condition $y(0)=1$.

We use the method of Laplace transforms. We also recognize that

$\int_0^t e^{-2\tau} y(t-\tau)\,d\tau$ is the convolution product of $f(t)=e^{-2t}$ and $y=y(t)$

Therefore

$$\mathcal{L}\{y'\}(s) = \mathcal{L}\{1 - f*y\}(s)$$

$$s\mathcal{L}\{y\}(s) - y(0) = \frac{1}{s} - \mathcal{L}\{f\}(s)\cdot\mathcal{L}\{y\}(s)$$ (by formulas 6, 1, and 12 in the short table)

$$s\mathcal{L}\{y\}(s) - 1 = \frac{1}{s} - \frac{1}{s+2}\cdot\mathcal{L}\{y\}(s)$$ (by formula 2 with $a=-2$)

$$\Rightarrow \left(s+\frac{1}{s+2}\right)\mathcal{L}\{y\}(s) = 1+\frac{1}{s}.$$

Multiplying through by $s(s+2)$ yields

$$s\big(\underbrace{s(s+2)+1}_{s^2+2s+1}\big)\mathcal{L}\{y\}(s) = (s+1)(s+2),$$

$$\Rightarrow \mathcal{L}\{y\}(s) = \frac{(s+1)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)} = \frac{A}{s}+\frac{B}{s+1}.$$

Then $s+2 = s(s+1)\left[\frac{A}{s}+\frac{B}{s+1}\right]$

$$\Rightarrow s+2 = A(s+1)+Bs.$$

Set $s=0$ to find A: $2=A$. Set $s=-1$ to find B: $1=-B$.

Thus $y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s}-\frac{1}{s+1}\right\}$

$$\Rightarrow \boxed{y(t) = 2-e^{-t}}$$ (by formulas 1 and 2 (with $a=-1$)).

3.[20] Find the solution of the system

$$\frac{dx}{dt} = x + 6y$$

$$\frac{dy}{dt} = x - 4y$$

that satisfies $x(0) = 5$, $y(0) = 9$.

The vector-matrix formulation: $\vec{x}' = \overbrace{\begin{bmatrix} 1 & 6 \\ 1 & -4 \end{bmatrix}}^{A} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$.

$\vec{x} = \vec{k} e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda \vec{k} = A\vec{k}$. Then λ satisfies

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 6 \\ 1 & -4-\lambda \end{vmatrix} = (4+\lambda)(\lambda-1) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2$$

Eigen values	Eigenvectors
$\lambda_1 = 2$	$\vec{k}^{(1)} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$
$\lambda_2 = -5$	$\vec{k}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigenvector corresponding to $\lambda = 2$:

$$(A - \lambda I)\vec{k} = \vec{0}$$

$$\begin{bmatrix} 1-2 & 6 \\ 1 & -4-2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -k_1 + 6k_2 = 0 \\ \cancel{k_1 - 6k_2 = 0} \text{ Redundant} \end{cases}$$

$$\Rightarrow k_1 = 6k_2 \Rightarrow \vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 6 \\ 1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = -5$: $(A - \lambda I)\vec{k} = \vec{0} \Rightarrow \begin{bmatrix} 1-(-5) & 6 \\ 1 & -4-(-5) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} \cancel{6k_1 + 6k_2 = 0} \text{ Redundant} \\ k_1 + k_2 = 0 \Rightarrow k_2 = -k_1 \end{cases} \therefore \vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solutions: $\vec{x}^{(1)} = \vec{k}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} e^{2t}$, $\vec{x}^{(2)} = \vec{k}^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}$.

$$W\left(\vec{x}^{(1)}\ \vec{x}^{(2)}\right) = \begin{vmatrix} 6e^{2t} & e^{-5t} \\ e^{2t} & -e^{-5t} \end{vmatrix} = -6e^{-3t} - e^{-3t} = -7e^{-3t} \neq 0.$$

Therefore $\begin{bmatrix} 6 \\ 1 \end{bmatrix} e^{2t}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}$ forms a fundamental set of solutions. The general solution is $\vec{x}(t) = c_1 \begin{bmatrix} 6 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}$ where c_1 and c_2 are arbitrary constants

$$\begin{bmatrix} 5 \\ 9 \end{bmatrix} \overset{\text{Want}}{=} \vec{x}(0) = c_1 \begin{bmatrix} 6 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$ Therefore $\begin{bmatrix} 6 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-7} \begin{bmatrix} -1 & -1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -14 \\ 49 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}.$$ Thus $\boxed{\vec{x}(t) = 2\begin{bmatrix} 6 \\ 1 \end{bmatrix} e^{2t} - 7\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}}$.

4.[19] Find the general solution of the system $\mathbf{X}' = \overbrace{\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}}^{A}\mathbf{X}$.

1 pt. to here $\vec{x} = \vec{k}e^{\lambda t}$ in $\vec{x}' = A\vec{x}$ leads to $\lambda\vec{k} = A\vec{k}$. Then λ satisfies

3 pts. to here $0 = \det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix} = (6+\lambda)(\lambda-4) + 25 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$.

5 pts. to here $\therefore$ $\lambda = -1$ (multiplicity two). An eigenvector corresponding to $\lambda = -1$ satisfies

$$(A - \lambda I)\vec{k} = \vec{0} \quad \text{or} \quad \begin{bmatrix} -6-(-1) & 5 \\ -5 & 4-(-1) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{cases} -5k_1 + 5k_2 = 0 \\ \cancel{-5k_1 + 5k_2 = 0} \text{ Redundant} \end{cases}$$

$\therefore$ $k_2 = k_1$ so $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We take $k_1 = 1$ so $\vec{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. 7 pts. to here

Because there is no second linearly independent eigenvector corresponding to the repeated eigenvalue $\lambda = -1$, we consider a second solution of the form $\vec{x} = \vec{k}te^{\lambda t} + \vec{l}e^{\lambda t}$. 9 pts. to here.
Substituting this in $\vec{x}' = A\vec{x}$ we find that $\vec{k}$, $\vec{l}$, and λ must satisfy

$(A - \lambda I)\vec{k} = \vec{0}$ and $(A - \lambda I)\vec{l} = \vec{k}$. But from above $\vec{k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = -1$.

11 pts. to here Therefore $(A - \lambda I)\vec{l} = \vec{k}$ becomes $\begin{bmatrix} -6-(-1) & 5 \\ -5 & 4-(-1) \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\begin{cases} -5l_1 + 5l_2 = 1 \\ \cancel{-5l_1 + 5l_2 = 1} \end{cases}$ Redundant.

$\therefore$ $l_2 = \frac{1}{5} + l_1$ and $\vec{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} l_1 \\ \frac{1}{5} + l_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} + l_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We may

13 pts. to here safely take $l_1 = 0$ so $\vec{l} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$. Thus

17 pts. to here $\vec{x}^{(1)} = \vec{k}e^{\lambda t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$ and $\vec{x}^{(2)} = \vec{k}te^{\lambda t} + \vec{l}e^{\lambda t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{-t} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}e^{-t}$ are solutions

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} e^{-t} & te^{-t} \\ e^{-t} & te^{-t} + \frac{1}{5}e^{-t} \end{vmatrix} = te^{-2t} + \frac{1}{5}e^{-2t} - te^{-2t} = \frac{1}{5}e^{-2t} \neq 0.$$

Therefore $\vec{x}^{(1)}, \vec{x}^{(2)}$ is a fundamental set of solutions to $\vec{x}' = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}\vec{x}$ and the

19 pts. to here general solution is $\boxed{\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}te^{-t} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}e^{-t}\right)}$ where c_1 and c_2 are arbitrary constants.

5.[20] (a) If $\Phi(t)$ is a fundamental matrix for $\mathbf{X}' = \mathbf{AX}$, what is the general solution of $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$?

$$\boxed{\vec{X}(t) = \Phi(t)\vec{c} + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\vec{F}(s)\,ds}$$

where $\vec{c}$ is an arbitrary constant vector and t_0 is any fixed point in the interval on which we are solving the system.

(b) Given that $\Phi(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ is a fundamental matrix for the homogeneous system $\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{X}$,

find the general solution of the nonhomogeneous system $\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sec(t) \\ 0 \end{pmatrix}$.

$$\Phi^{-1}(s) = \frac{1}{\det \Phi(s)}\begin{bmatrix} \varphi_{22} & -\varphi_{12} \\ -\varphi_{21} & \varphi_{11} \end{bmatrix} = \frac{1}{1}\begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix}$$

$$\Phi(t)\int^{t} \Phi^{-1}(s)\vec{F}(s)\,ds = \Phi(t)\int^{t}\begin{bmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{bmatrix}\begin{bmatrix} \sec(s) \\ 0 \end{bmatrix} ds = \Phi(t)\int^{t}\begin{bmatrix} 1 \\ -\tan(s) \end{bmatrix} ds = \Phi(t)\begin{bmatrix} t \\ \ln|\cos(t)| \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} t \\ \ln|\cos(t)| \end{bmatrix} = t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + \ln|\cos(t)|\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

$$\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t) = c_1\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + c_2\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + t\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + \ln|\cos(t)|\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

$$\boxed{\vec{x}(t) = (c_1 + t)\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + \left(c_2 + \ln|\cos(t)|\right)\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}}.$$

SHORT TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. e^{at}	$\dfrac{1}{s-a}$
3. t^n	$\dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3....$
4. $\sin(kt)$	$\dfrac{k}{s^2+k^2}$
5. $\cos(kt)$	$\dfrac{s}{s^2+k^2}$
6. $f'(t)$	$sF(s) - f(0)$
7. $f''(t)$	$s^2F(s) - sf(0) - f'(0)$
8. $\delta(t-t_0)$	e^{-st_0}
9. $e^{at}f(t)$	$F(s-a)$
10. $\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
11. $f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
12. $\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$