Bounded Linear Functionals on $C[0,1]$ (Not in Rudin)

<u>Definition</u>: Let $(\mathbb{X}, \|\cdot\|)$ be a (real) normed linear space. We say that a function $\Lambda : \mathbb{X} \to \mathbb{R}$ is a <u>bounded</u> (or <u>continuous</u>) <u>linear functional</u> on $\mathbb{X}$ provided:

(1) $\Lambda(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\Lambda(\vec{v}_1) + c_2\Lambda(\vec{v}_2)$ for all $c_1, c_2 \in \mathbb{R}$ and all $\vec{v}_1, \vec{v}_2 \in \mathbb{X}$;

(2) there exists a real number K such that $|\Lambda(\vec{v})| \le K\|\vec{v}\|$ for all $\vec{v} \in \mathbb{X}$.

<u>Examples of bounded linear functionals</u>.

① Consider $\mathbb{X} = \mathbb{R}^n$ with norm $\|(x_1, \ldots, x_n)\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$.

If $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ define $\Lambda_z : \mathbb{R}^n \to \mathbb{R}$ by

$$\Lambda_z(x) = x \cdot z \quad (\text{inner product of } x \text{ with } z)$$
$$= x_1 z_1 + \ldots + x_n z_n .$$

It is easy to see that Λ_z is linear:

$$\Lambda_z(\alpha x + \beta y) = (\alpha x + \beta y)\cdot z = \alpha\, x\cdot z + \beta y \cdot z = \alpha \Lambda_z(x) + \beta \Lambda_z(y).$$

The Cauchy-Schwarz inequality (Thm 1.37(d)) shows that Λ_z is bounded:

$$|\Lambda_z(x)| = |x \cdot z| \le \sqrt{x_1^2 + \ldots + x_n^2}\sqrt{z_1^2 + \ldots + z_n^2} = \overset{K}{\|z\|_2}\|x\|_2$$

for all $x \in \mathbb{R}^n$.

② Consider $\mathbb{X} = C[0,1]$ with norm $\|f\|_2 = \left(\int_0^1 [f(x)]^2 dx\right)^{1/2}$.

If $g \in \mathscr{R}$ on $[0,1]$, define $\Lambda_g : C[0,1] \to \mathbb{R}$ by

$$\Lambda_g(f) = \int_0^1 f(x) g(x)\, dx \qquad (f \in C[0,1]).$$

Clearly Λ_g is linear: $\Lambda_g(\alpha f+\beta h)=\alpha\Lambda_g(f)+\beta\Lambda_g(h)$. The Cauchy-Schwarz inequality (#10(c), p.139 with $p=q=2$) shows that Λ_g is bounded:

$$\begin{aligned}|\Lambda_g(f)| &= \left|\int_0^1 f(x)g(x)\,dx\right| \\ &\le \left(\int_0^1 [f(x)]^2dx\right)^{1/2}\left(\int_0^1[g(x)]^2dx\right)^{1/2} \\ &= \underbrace{\|g\|_2}_{K}\|f\|_2\end{aligned}$$

for all $f\in C[0,1]$.

(3) Consider $X = C[0,1]$ with the uniform norm $\|f\|_u = \sup\{|f(x)|: x\in[0,1]\}$. Recall that if $f\in C[0,1]$ and $\alpha\in BV[0,1]$ then

$$(\dagger)\qquad \left|\int_0^1 f\,d\alpha\right| \le \|f\|_u \underbrace{Var(\alpha;0,1)}_{K}.$$

The linearity of the Riemann-Stieltjes integral shows that each $\alpha\in BV[0,1]$ gives rise to a linear functional Λ_α on $C[0,1]$ via

$$\Lambda_\alpha(f)=\int_0^1 f\,d\alpha \qquad (f\in C[0,1]).$$

The inequality $(\dagger)$ shows that Λ_α is bounded on $(C[0,1],\|\cdot\|_u)$.

The converse of this result is also true.

<u>The Riesz Representation Theorem</u> (F. Riesz 1909): Let Λ be a bounded linear functional on $(C[0,1],\|\cdot\|_u)$. Then there exists $\alpha\in BV[0,1]$ such that $\Lambda(f)=\int_0^1 f\,d\alpha$ for all $f\in C[0,1]$.

<u>Lemma 1</u> (Helly's Selection Theorem; cf. #13(a) p. 167 of Rudin)

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of increasing functions on $[a,b]$ such that

Could have m and M here.

$0 \le f_n(x) \le 1$ for all $x \in [a,b]$ and all $n \ge 1$. Then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which is pointwise convergent on $[a,b]$.

Know statement and proof.

<u>Proof</u>: Let $E = Q \cap [a,b]$. $\boxed{\text{Theorem 7.23}}$ implies the existence of a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ converges for each $x \in E$, say

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \qquad (x \in E).$$

It is clear that f is increasing on E. Define a function on $[a,b]$ by

$$F(y) = \sup\{ f(x) : x \in E, x \le y \}.$$

It is easy to see that $F(y) = f(y)$ for all $y \in E$ and that F is increasing on $[a,b]$. The claim that if $y \in (a,b) \setminus D(F)$ then $F(y) = \lim_{k\to\infty} f_{n_k}(y)$.

To prove the claim, fix $y \in (a,b) \setminus D(F)$ and let $\varepsilon > 0$. Choose $x, z \in E$ with the following properties

(i) $z < y < x$;

(ii) $f(z) > F(y) - \frac{\varepsilon}{2}$; Def. of F

(iii) $f(x) < F(y) + \frac{\varepsilon}{2}$. Continuity of F at y, $f = F|_E$, and density of E in (a,b).

Then choose positive integers $K_1 = K_1(z,\varepsilon)$ and $K_2 = K_2(x,\varepsilon)$ such that

(iv) $|f(z) - f_{n_k}(z)| < \frac{\varepsilon}{2}$ for all $k \ge K_1$, and

(v) $|f(x) - f_{n_k}(x)| < \frac{\varepsilon}{2}$ for all $k \ge K_2$.

If $k \geq K \equiv \max\{K_1, K_2\}$ then

$$F(y) - f_{n_k}(y) \overset{\text{(i) and } f_{n_k}\uparrow}{\leq} F(y) - f_{n_k}(z) = F(y) - f(z) + f(z) - f_{n_k}(z) \overset{\text{(ii) and (iv)}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and

$$F(y) - f_{n_k}(y) \overset{\text{(i) and } f_{n_k}\uparrow}{\geq} F(y) - f_{n_k}(x) = F(y) - f(x) + f(x) - f_{n_k}(x) \overset{\text{(iii) and (v)}}{>} -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon .$$

I.e. $|F(y) - f_{n_k}(y)| < \varepsilon$ for all $k \geq K$, and the claim is established

Since F is increasing on $[a,b]$, $D(F)$ is at most countable. Theorem 7.23 then guarantees the existence of a subsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{f_{n_{k_j}}(x)\}_{j=1}^{\infty}$ converges for all $x \in \{a,b\} \cup D(F)$.

Define

$$\tilde{F}(y) = \begin{cases} \lim_{j\to\infty} f_{n_{k_j}}(y) & \text{if } y \in \{a,b\} \cup D(F), \\ F(y) & \text{if } y \in (a,b) \setminus D(F). \end{cases}$$

Clearly $f_{n_{k_j}} \longrightarrow \tilde{F}$ pointwise on $[a,b]$. QED.

<u>Lemma 2</u>: Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions in $BV[a,b]$. If there exist real numbers M_1 and M_2 such that

(i) $Var(g_n; a, b) \leq M_1$ for all $n \geq 1$, and

(ii) $|g_n(a)| \leq M_2$ for all $n \geq 1$,

then there exists a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ which converges pointwise on $[a,b]$.

Proof : A in HW set #7 .

<u>Lemma 3</u>. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence in $BV[a,b]$ such that $g_n \to g$ pointwise on $[a,b]$. If there exists a real number M such that $Var(g_n; a,b) \le M$ for all $n \ge 1$ then $g \in BV[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f\,dg_n = \int_a^b f\,dg$$

for all $f \in C[a,b]$.

<u>Proof</u>: Observe that for any partition $P: a = x_0 < x_1 < \dots < x_m = b$ of $[a,b]$ we have

$$\sum_{k=1}^{m} |g_n(x_k) - g_n(x_{k-1})| \le Var(g_n; a,b) \le M.$$

Since $g_n \to g$ pointwise on $[a,b]$, taking the limit as $n \to \infty$ of the LHS gives

$$\sum_{k=1}^{m} |g(x_k) - g(x_{k-1})| \le M.$$

Consequently, $g \in BV[a,b]$.

Fix $f \in C[a,b]$ and let $\varepsilon > 0$. Since f is uniformly continuous on $[a,b]$, there is a partition $a = x_0 < x_1 < \dots < x_m = b$ of $[a,b]$ such that

$$\sup_{x,y \in [x_{k-1}, x_k]} |f(x) - f(y)| < \frac{\varepsilon}{3M} \quad \text{for all } 1 \le k \le m.$$

Next, choose an integer $N = N(\varepsilon, x_0, \dots, x_m) \ge 1$ such that

$$|g(x_k) - g_n(x_k)| < \frac{\varepsilon}{6m\|f\|_u}$$

for all $0 \le k \le m$ and all $n \ge N$. Note that

$$\int_a^b f\,dg_n = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} f\,dg_n$$

$$= \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \left(f(x) - f(x_{k-1})\right) dg_n + \sum_{k=1}^{m} f(x_{k-1}) \int_{x_{k-1}}^{x_k} dg_n$$

$$= \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \left(f(x) - f(x_{k-1})\right) dg_n + \sum_{k=1}^{m} f(x_{k-1}) \left(g_n(x_k) - g_n(x_{k-1}) \right).$$

Similarly $\displaystyle \int_a^b f\,dg = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \left(f(x) - f(x_{k-1})\right) dg + \sum_{k=1}^{m} f(x_{k-1}) \left(g(x_k) - g(x_{k-1}) \right).$

Therefore, for all $n \geq N$, *we have*

$$\left| \int_a^b f\,dg - \int_a^b f\,dg_n \right| \leq \sum_{k=1}^{m} \left| \int_{x_{k-1}}^{x_k} (f(x) - f(x_{k-1}))\,dg \right| + \sum_{k=1}^{m} \left| \int_{x_{k-1}}^{x_k} (f(x) - f(x_{k-1}))\,dg_n \right|$$

$$+ \sum_{k=1}^{m} |f(x_{k-1})| \left[|g(x_k) - g_n(x_k)| + |g(x_{k-1}) - g_n(x_{k-1})| \right]$$

$$\leq \sum_{k=1}^{m} \frac{\varepsilon}{3M} \operatorname{Var}(g; x_{k-1}, x_k) + \sum_{k=1}^{m} \frac{\varepsilon}{3M} \operatorname{Var}(g_n; x_{k-1}, x_k)$$

$$+ \sum_{k=1}^{m} \|f\|_u \left[\frac{\varepsilon}{6m\|f\|_u} + \frac{\varepsilon}{6m\|f\|_u} \right]$$

$$= \frac{\varepsilon}{3M} \operatorname{Var}(g; a, b) + \frac{\varepsilon}{3M} \operatorname{Var}(g_n; a, b) + \frac{\varepsilon}{3}$$

$$\leq \varepsilon,$$

Q.E.D.

Proof of the Riesz Representation Theorem: We need to find $\alpha \in BV[0,1]$ such that $\Lambda(f) = \int_0^1 f\,d\alpha$ for all $f \in C[0,1]$. Recall that if

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\tfrac{k}{n}\right) x^k (1-x)^{n-k} \quad \text{for } 0 \le x \le 1 \text{ and } n = 1,2,3,\ldots \text{ then}$$

$B_n(f) \to f$ uniformly on $[0,1]$ for each $f \in C[0,1]$. We know that there exists a real number K, independent of f in $C[0,1]$, such that

$$(\dagger) \qquad |\Lambda(f) - \Lambda(B_n(f))| = |\Lambda(f - B_n(f))| \le K \|f - B_n(f)\|_u \to 0 \text{ as } n \to \infty.$$

It follows that for all $f \in C[0,1]$ our candidate $\alpha = \alpha(x)$ must satisfy:

$$(*) \qquad \int_0^1 f\,d\alpha = \Lambda(f) = \lim_{n\to\infty} \Lambda(B_n(f)) = \lim_{n\to\infty} \Lambda\left(\sum_{k=0}^{n} \binom{n}{k} f\left(\tfrac{k}{n}\right) (\cdot)^k (1-\cdot)^{n-k}\right)$$

$$\left.\begin{aligned} &= \lim_{n\to\infty} \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \overbrace{\binom{n}{k} \Lambda\left((\cdot)^k (1-\cdot)^{n-k}\right)}^{\text{jump in } \alpha_n \text{ at } x = k/n} \\ &= \lim_{n\to\infty} \int_0^1 f\,d\alpha_n \end{aligned}\right\} (\text{cf. Thm. 6.16})$$

where $\alpha_n(x) = I(x)\Lambda(1) + \sum_{k=1}^{n} H\left(x - \tfrac{k}{n}\right) \binom{n}{k} \Lambda\left((\cdot)^k (1-\cdot)^{n-k}\right) \qquad (n = 1,2,3,\ldots)$.

[Note that α_n is constant on each interval $\left(\frac{k-1}{n}, \frac{k}{n}\right)$ for $1 \le k \le n$ and the jump in α_n at $x = \frac{k}{n}$ $(0 \le k \le n)$ is

$$\alpha_n\left(\tfrac{k^+}{n}\right) - \alpha_n\left(\tfrac{k^-}{n}\right) = \binom{n}{k} \Lambda\left((\cdot)^k (1-\cdot)^{n-k}\right)$$

so

$$(**) \qquad \int_0^1 f\,d\alpha_n \overset{\text{Thm 6.16}}{=} \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} \Lambda\left((\cdot)^k (1-\cdot)^{n-k}\right) \overset{\text{linearity of } \Lambda}{=} \Lambda(B_n(f)) \,.\,]$$

Therefore, examining (*) in light of Lemma 3, it follows that a candidate for α is the pointwise limit

$$\alpha(x) = \lim_{n\to\infty} \alpha_n(x) \qquad (0 \le x \le 1)$$

It turns out that this is not quite achievable because the pointwise limit in the right member of the preceding displayed equation may not exist. However the limit does exist for an appropriately chosen subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$ and this suffices as we now show.

<u>Claim</u>: $\mathrm{Var}(\alpha_n; 0,1) \le K$ for all $n = 1, 2, 3, \ldots$
(Here K is any real number such that $|\Lambda(f)| \le K\|f\|_u$ for all f in $C[0,1]$.)

<u>Proof of Claim</u>: Fix a positive integer n. Since α_n is piecewise constant there exist signs $\varepsilon_k \in \{-1,1\}$ for $0 \le k \le n$ such that

$$\mathrm{Var}(\alpha_n; 0,1) = \sup\left\{ \sum_{j=1}^{M} |\alpha_n(x_j) - \alpha_n(x_{j-1})| : 0 = x_0 < x_1 < \ldots < x_M = 1 \text{ is any partition of } [0,1] \right\}$$

$$= \sum_{k=0}^{n} \left| \Lambda\left(\binom{n}{k} (\cdot)^k (1-\cdot)^{n-k} \right) \right|$$

$$= \sum_{k=0}^{n} \varepsilon_k \Lambda\left(\binom{n}{k} (\cdot)^k (1-\cdot)^{n-k} \right)$$

$$= \Lambda\left(\sum_{k=0}^{n} \varepsilon_k \binom{n}{k} (\cdot)^k (1-\cdot)^{n-k} \right)$$

$$\le K \sup\left\{ \left| \sum_{k=0}^{n} \varepsilon_k \binom{n}{k} x^k (1-x)^{n-k} \right| : 0 \le x \le 1 \right\}$$

$$\leq K \sup\left\{ \sum_{k=0}^{n} \left| \varepsilon_k \binom{n}{k} x^k (1-x)^{n-k} \right| : 0 \leq x \leq 1 \right\}$$

$$= K \sup\left\{ \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{\text{This is } B_n(1) \equiv 1 \text{ for all } 0 \leq x \leq 1} : 0 \leq x \leq 1 \right\}$$

$$= K. \qquad \text{(Q.E.D. for claim .)}$$

(Back to proof of R.R.T.) Note that $\alpha_n(0) = 0$ for all $n \geq 1$. Therefore the claim and Lemma 2 imply the existence of a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$ which converges pointwise on $[0,1]$, say

$$\alpha(x) = \lim_{k\to\infty} \alpha_{n_k}(x) \qquad (0 \leq x \leq 1).$$

Lemma 3 guarantees that $\alpha \in BV[0,1]$ and

$$(\blacksquare) \qquad \lim_{k\to\infty} \int_0^1 f \, d\alpha_{n_k} = \int_0^1 f \, d\alpha \qquad \left(f \in C[0,1] \right)$$

Using $(\dagger)$, $(**)$, and $(\blacksquare)$ yields

$$\Lambda(f) = \lim_{k\to\infty} \Lambda\left(B_{n_k}(f) \right) = \lim_{k\to\infty} \int_0^1 f \, d\alpha_{n_k} = \int_0^1 f \, d\alpha$$

for all f in $C[0,1]$. Q.E.D.

Notes: If $(\mathfrak{X}, \|\cdot\|)$ is a Banach space then the <u>dual space</u> of $\mathfrak{X}$ is defined to be the vector space of all bounded linear functionals $\Lambda : \mathbb{X} \to \mathbb{R}$. We denote the dual space of $\mathbb{X}$ by $\mathbb{X}^*$. It is not hard to see that $\mathbb{X}^*$ is a Banach space in its own right via the norm

$$(\dagger) \qquad \| \Lambda \| = \sup \left\{ |\Lambda(\vec{x})| : \vec{x} \in \mathbb{X}, \ \|\vec{x}\| = 1 \right\}.$$

(Cf. Proposition 10.3, Royden, p. 221.) The Riesz Representation Theorem shows that $(C[a,b], \|\cdot\|_u)^*$ can be identified with $\overbrace{BV[a,b]}^{\text{a subspace of}}$.

Q: What is the Banach space norm $(\dagger)$ of the bounded linear functional Λ_α on $C[a,b]$ given by

$$\Lambda_\alpha(f) = \int_a^b f \, d\alpha$$

where $\alpha \in BV[a,b]$?

A: $\|\Lambda_\alpha\| = Var(\alpha; a, b)$ which defines a norm on the subspace of $BV[a,b]$ consisting of those functions $\alpha \in BV[a,b]$ satisfying $\alpha(a) = 0$. Compare this norm with the Banach space norm

$$N_2(\alpha) = |\alpha(a)| + Var(\alpha; a, b)$$

on $BV[a,b]$. (See HW Set #5, problem C.)