Bounded Linear Functionals on $C[0,1]$ (Not in Rudin)

Definition: Let $(X, \| \cdot \|)$ be a (real) normed linear space. We say that a function $\Lambda : X \rightarrow \mathbb{R}$ is a bounded (or continuous) linear functional on $X$ provided:

1. $\Lambda(c_1 \overline{v}_1 + c_2 \overline{v}_2) = c_1 \Lambda(\overline{v}_1) + c_2 \Lambda(\overline{v}_2)$ for all $c_1, c_2 \in \mathbb{R}$ and all $\overline{v}_1, \overline{v}_2 \in X$.
2. There exists a real number $K$ such that $|\Lambda(\overline{v})| \leq K \| \overline{v} \|$ for all $\overline{v} \in X$.

Examples of bounded linear functionals.

1. Consider $X = \mathbb{R}^n$ with norm $\| (x_1, \ldots, x_n) \|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$.

If $\overline{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ define $\Lambda_{\overline{z}} : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\Lambda_{\overline{z}}(\overline{x}) = \overline{x} \cdot \overline{z} \quad \text{(inner product of } \overline{x} \text{ with } \overline{z})$$

$$= x_1 z_1 + \ldots + x_n z_n.$$  

It is easy to see that $\Lambda_{\overline{z}}$ is linear:

$$\Lambda_{\overline{z}}(\alpha \overline{x} + \beta \overline{y}) = (\alpha \overline{x} + \beta \overline{y}) \cdot \overline{z} = \alpha \overline{x} \cdot \overline{z} + \beta \overline{y} \cdot \overline{z} = \alpha \Lambda_{\overline{z}}(\overline{x}) + \beta \Lambda_{\overline{z}}(\overline{y}).$$

The Cauchy–Schwarz inequality (Thm 1.37(a)) shows that $\Lambda_{\overline{z}}$ is bounded:

$$|\Lambda_{\overline{z}}(\overline{x})| = | \overline{x} \cdot \overline{z} | \leq \sqrt{x_1^2 + \ldots + x_n^2} \sqrt{z_1^2 + \ldots + z_n^2} = \| \overline{x} \|_2 \| \overline{z} \|_2$$

for all $\overline{x} \in \mathbb{R}^n$.

2. Consider $X = C[0,1]$ with norm $\| f \|_2 = \left( \int_0^1 |f(x)|^2 \, dx \right)^{1/2}$.

If $g \in \mathbb{R}$ on $[0,1]$, define $\Lambda_g : C[0,1] \rightarrow \mathbb{R}$ by

$$\Lambda_g(f) = \int_0^1 f(x)g(x) \, dx \quad (f \in C[0,1]).$$
Clearly $\Lambda_g$ is linear: $\Lambda_g(\alpha f + \beta h) = \alpha \Lambda_g(f) + \beta \Lambda_g(h)$. The Cauchy-Schwarz inequality (#10(c), p.139 with $p=q=2$) shows that $\Lambda_g$ is bounded:

$$|\Lambda_g(f)| = \left| \int_0^1 f(x)g(x)\,dx \right|$$

$$\leq \left( \int_0^1 [f(x)]^2\,dx \right)^{1/2} \left( \int_0^1 [g(x)]^2\,dx \right)^{1/2}$$

$$= \|g\|_2 \|f\|_2$$

for all $f \in C[0,1]$.

3. Consider $X = C[0,1]$ with the uniform norm

$$\|f\|_u = \sup \{|f(x)| : x \in [0,1]\}.$$  Recall that if $f \in C[0,1]$ and $\alpha \in BV[0,1]$ then

$$(\dagger) \quad \left| \int_0^1 f(x)\,dx \right| \leq \|f\|_u \text{Var}(\alpha; [0,1]).$$

The linearity of the Riemann-Stieltjes integral shows that each $\alpha \in BV[0,1]$ gives rise to a linear functional $\Lambda_\alpha$ on $C[0,1]$ via

$$\Lambda_\alpha(f) = \int_0^1 f(x)\,d\alpha \quad (f \in C[0,1]).$$

The inequality $(\dagger)$ shows that $\Lambda_\alpha$ is bounded on $(C[0,1], \|\cdot\|_u)$. The converse of this result is also true.

The Riesz Representation Theorem (F. Riesz 1909): Let $\Lambda$ be a bounded linear functional on $(C[0,1], \|\cdot\|_u)$. Then there exists $\alpha \in BV[0,1]$ such that $\Lambda(f) = \int_0^1 f\,d\alpha$ for all $f \in C[0,1]$. 
Lemma 1 (Kelley's Selection Theorem; cf. #13(a) p. 167 of Rudin)

Let \( \{ f_n \}_{n=1}^{\infty} \) be a sequence of increasing functions on \([a,b]\) such that

\( 0 \leq f_n(x) \leq M \) for all \( x \in [a,b] \) and all \( n \geq 1 \). Then there exists a subsequence

\( \{ f_{n_k} \}_{k=1}^{\infty} \)

which is pointwise convergent on \([a,b]\).

**Proof:** Let \( E = \mathbb{Q} \cap [a,b] \). Theorem 7.23 implies the existence of a subsequence \( \{ f_{n_k} \}_{k=1}^{\infty} \) of \( \{ f_n \}_{n=1}^{\infty} \) such that \( \{ f_{n_k}(x) \}_{k=1}^{\infty} \) converges for each \( x \in E \), say

\[ f(x) = \lim_{k \to \infty} f_{n_k}(x) \quad (x \in E). \]

It is clear that \( f \) is increasing on \( E \). Define a function on \([a,b]\) by

\[ F(y) = \sup \{ f(x) : x \in E, x \leq y \}. \]

It is easy to see that \( F(y) = f(y) \) for all \( y \in E \) and that \( F \) is increasing on \([a,b]\). The claim that if \( y \in (a,b) \setminus D(F) \) then \( F(x) = \lim_{k \to \infty} f_{n_k}(y) \).

To prove the claim, fix \( y \in (a,b) \setminus D(F) \) and let \( \varepsilon > 0 \). Choose \( x, z \in E \) with the following properties

1. \( z < y < x \); 
2. \( f(z) > F(y) - \frac{\varepsilon}{2} \); 
3. \( f(x) < F(y) + \frac{\varepsilon}{2} \).

Then choose positive integers \( K_1 = K_1(z, \varepsilon) \) and \( K_2 = K_2(x, \varepsilon) \) such that

4. \( |f(z) - f_{n_k}(z)| < \frac{\varepsilon}{2} \) for all \( k \geq K_1 \), and
5. \( |f(x) - f_{n_k}(x)| < \frac{\varepsilon}{2} \) for all \( k \geq K_2 \).
If \( k > K = \max \{ K_1, K_2 \} \) then

\[(ii) \text{ and } (iv)\]
\[F(y) - f_k(y) \leq F(y) - f_k(x) = F(y) - f(x) + f(x) - f_k(x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

and

\[(i) \text{ and } f_k \downarrow\]
\[F(y) - f_k(y) \geq F(y) - f_k(x) = F(y) - f(x) + f(x) - f_k(x) \geq -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon.\]

I.e. \(|F(y) - f_k(y)| < \varepsilon\) for all \( k \geq K \), and the claim is established.

Since \( F \) is increasing on \([a, b]\), \( D(F) \) is at most countable. Theorem 7.23 then guarantees the existence of a subsequence \( \{ f_{k_j} \}_{j=1}^{\infty} \) of \( \{ f_k \}_{k=1}^{\infty} \) such that \( \{ f_{k_j}(x) \}_{j=1}^{\infty} \) converges for all \( x \in [a, b] \setminus D(F) \).

Define

\[\widetilde{F}(y) = \begin{cases} 
\lim_{j \to \infty} f_{k_j}(y) & \text{ if } y \in [a, b] \setminus D(F), \\
F(y) & \text{ if } y \in (a, b) \setminus D(F).
\end{cases}\]

Clearly \( f_{k_j} \to \widetilde{F} \) pointwise on \([a, b]\). QED.

**Lemma 2:** Let \( \{ g_n \}_{n=1}^{\infty} \) be a sequence of functions in \( BV[a, b] \). If there exist real numbers \( M_1 \) and \( M_2 \) such that

(i) \( \text{Var}(g_n; [a, b]) \leq M_1 \) for all \( n \geq 1 \), and

(ii) \( |g_n(a)| \leq M_2 \) for all \( n \geq 1 \),

then there exists a subsequence \( \{ g_{n_k} \}_{k=1}^{\infty} \) which converges pointwise on \([a, b]\).

**Proof:** As in HW Set #7.
Lemma 3. Let \(\{g_n\}_{n=1}^{\infty}\) be a sequence in \(BV[a,b]\) such that \(g_n \to g\) pointwise on \([a,b]\). If there exists a real number \(M\) such that \(\text{Var}(g_n; a,b) \leq M\) for all \(n \geq 1\), then \(g \in BV[a,b]\) and

\[
\lim_{n \to \infty} \int_a^b f \, dg_n = \int_a^b f \, dg
\]

for all \(f \in C[a,b]\).

**Proof:** Observe that for any partition \(P: a = x_0 < x_1 < \ldots < x_m = b\) of \([a,b]\) we have

\[
\sum_{k=1}^{m} |g_n(x_k) - g_n(x_{k-1})| \leq \text{Var}(g_n; a,b) \leq M.
\]

Since \(g_n \to g\) pointwise on \([a,b]\), taking the limit as \(n \to \infty\) of the LHS gives

\[
\sum_{k=1}^{m} |g(x_k) - g(x_{k-1})| \leq M.
\]

Consequently, \(g \in BV[a,b]\).

Fix \(f \in C[a,b]\) and let \(\varepsilon > 0\). Since \(f\) is uniformly continuous on \([a,b]\), there is a partition \(a = x_0 < x_1 < \ldots < x_m = b\) of \([a,b]\) such that

\[
\sup_{x,y \in [x_{k-1}, x_k]} |f(x) - f(y)| < \frac{\varepsilon}{3M}
\]

for all \(1 \leq k \leq m\).

Next, choose an integer \(N = N(\varepsilon, x_0, \ldots, x_m) \geq 1\) such that

\[
|g(x_k) - g_n(x_k)| < \frac{\varepsilon}{6m \|f\|_w}
\]

for all \(0 \leq k \leq m\) and all \(n \geq N\). Note that
\[
\begin{align*}
\int_a^b f(x) dg_n &= \sum_{k=1}^m \int_{x_{k-1}}^{x_k} f(x) dg_n \\
&= \sum_{k=1}^m \int_{x_{k-1}}^{x_k} (f(x) - f(x_{k-1})) dg_n + \sum_{k=1}^m f(x_{k-1}) \int_{x_{k-1}}^{x_k} dg_n \\
&= \sum_{k=1}^m \int_{x_{k-1}}^{x_k} (f(x) - f(x_{k-1})) dg_n + \sum_{k=1}^m f(x_{k-1}) (g_n(x_k) - g_n(x_{k-1}))
\end{align*}
\]

Similarly, \[
\begin{align*}
\int_a^b f(x) dg &= \sum_{k=1}^m \int_{x_{k-1}}^{x_k} (f(x) - f(x_{k-1})) dg + \sum_{k=1}^m f(x_{k-1}) (g(x_k) - g(x_{k-1}))
\end{align*}
\]

Therefore, for all \( n \geq N \), we have

\[
\begin{align*}
\left| \int_a^b f(x) dg - \int_a^b f(x) dg_n \right| &\leq \sum_{k=1}^m \left| \int_{x_{k-1}}^{x_k} (f(x) - f(x_{k-1})) dg \right| + \sum_{k=1}^m \left| \int_{x_{k-1}}^{x_k} g_n(x_k) - g_n(x_{k-1}) \right|
\end{align*}
\]

\[
\begin{align*}
&+ \sum_{k=1}^m |f(x_{k-1})| \left[ |g(x_k) - g(x_{k-1})| + |g(x_k) - g(x_{k-1})| \right]
\end{align*}
\]

\[
\begin{align*}
&\leq \sum_{k=1}^m \frac{\epsilon}{3M} \text{Var}(g; x_{k-1}, x_k) + \sum_{k=1}^m \frac{\epsilon}{3M} \text{Var}(g_n; x_{k-1}, x_k)
\end{align*}
\]

\[
\begin{align*}
&+ \sum_{k=1}^m \|f\|_u \left[ \frac{\epsilon}{6m \|f\|_u} + \frac{\epsilon}{6m \|f\|_u} \right]
\end{align*}
\]

\[
\begin{align*}
&= \frac{\epsilon}{3M} \text{Var}(g; a, b) + \frac{\epsilon}{3M} \text{Var}(g_n; a, b) + \frac{\epsilon}{3}
\end{align*}
\]

\[
\begin{align*}
\leq \epsilon, \quad \text{Q.E.D.}
\end{align*}
\]
Proof of the Riesz Representation Theorem: We need to find $\alpha \in BV[0,1]$ such that $\Lambda(f) = \int f \, d\alpha$ for all $f \in C[0,1]$. Recall that if

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k} \quad \text{for } 0 \leq x \leq 1 \text{ and } n=1,2,3,\ldots$$

then $B_n(f) \to f$ uniformly on $[0,1]$ for each $f \in C[0,1]$. We know that there exists a real number $K$, independent of $f$ in $C[0,1]$, such that

$$|\Lambda(f) - \Lambda(B_n(f))| = |\Lambda(f - B_n(f))| \leq K \|f - B_n(f)\|_u \to 0 \text{ as } n \to \infty.$$

It follows that our candidate $\alpha = \alpha(x)$ must satisfy:

$$\int \Lambda(f) \, d\alpha = \Lambda(f) = \lim_{n \to \infty} \Lambda(B_n(f)) = \lim_{n \to \infty} \Lambda \left( \sum_{k=0}^{n} \binom{n}{k} f(k/n) \cdot (1 - \cdot)^{n-k} \right)$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k} f(k/n) \Lambda((\cdot)^{n-k})$$

$$= \lim_{n \to \infty} \int f \, d\alpha_n \quad \text{(cf. Thm. 6.16)}$$

where $\alpha_n(x) = I(\cdot) \Lambda(\cdot) + \sum_{k=1}^{n} H(x - k/n) \binom{n}{k} \Lambda((\cdot)^{k}(1 - \cdot)^{n-k}) \quad (n=1,2,3,\ldots)$. Note that $\alpha_n$ is constant on each interval $(\frac{k-1}{n}, \frac{k}{n})$ for $1 \leq k \leq n$ and the jump in $\alpha_n$ at $x = \frac{k}{n}$ ($0 \leq k \leq n$) is

$$\alpha_n^+(\frac{k}{n}) - \alpha_n^-(\frac{k}{n}) = \binom{n}{k} \Lambda((\cdot)^{k}(1 - \cdot)^{n-k})$$

and

$$\int f \, d\alpha_n = \sum_{k=0}^{n} f(k/n) \binom{n}{k} \Lambda((\cdot)^{k}(1 - \cdot)^{n-k}) = \Lambda(B_n(f)) \quad \text{linearity of } \Lambda.$$
Therefore, examining (*) in light of Lemma 3, it follows that a candidate for $\lambda$ is the pointwise limit

$$\lambda(x) = \lim_{n \to \infty} \alpha_n(x) \quad (0 \leq x \leq 1)$$

pointwise.

It turns out that this is not quite achievable because the limit in the right member of the preceding displayed equation may not exist. However, the limit does exist for $\forall$ subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$, and this suffices so we now show.

Claim: $\text{Var}(\alpha_n; 0, 1) \leq K$ for all $n = 1, 2, 3, \ldots$

(Here $K$ is any real number such that $|\Lambda(f)| \leq K \|f\|_\infty$ for all $f \in C[0, 1]$.)

Proof of Claim: Fix a positive integer $n$. Since $\alpha_n$ is piecewise constant there exist signs $\varepsilon_k \in \{-1, 1\}$ for $0 \leq k \leq n$ such that

$$\text{Var}(\alpha_n; 0, 1) = \sup \left\{ \sum_{j=1}^{M} |\alpha_n(x_j) - \alpha_n(x_{j-1})| : 0 = x_0 < x_1 < \ldots < x_M = 1 \right\}
$$

is any partition of $[0, 1]$}

$$= \sum_{k=0}^{n} |\Lambda\left(\binom{n}{k}(\cdot)^{k}(1-\cdot)^{n-k}\right)|
$$

$$= \sum_{k=0}^{n} \varepsilon_k \Lambda\left(\binom{n}{k}(\cdot)^{k}(1-\cdot)^{n-k}\right)
$$

$$= \Lambda\left(\sum_{k=0}^{n} \varepsilon_k \binom{n}{k}(\cdot)^{k}(1-\cdot)^{n-k}\right)
$$

$$\leq K \sup\left\{ \left| \sum_{k=0}^{n} \varepsilon_k \binom{n}{k}(1-x)^{n-k} \right| : 0 \leq x \leq 1 \right\}.$$
\[ L = K \sup \left\{ \sum_{k=0}^{n} \left| \varepsilon_k \binom{n}{k} x^k (1-x)^{n-k} \right| : 0 \leq x \leq 1 \right\} \]

\[ = K \sup \left\{ \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} : 0 \leq x \leq 1 \right\} \]

This is \( B_n(1) = 1 \) for all \( 0 \leq x \leq 1 \)

\[ = K. \quad (\text{Q.E.D. for claim.}) \]

(Back to proof of R.R.T.) Note that \( \alpha_n(0) = 0 \) for all \( n \geq 1 \). Therefore the claim and Lemma 2 imply the existence of a subsequence \( \{ \alpha_{n_k} \}_{k=1}^{\infty} \) which converges pointwise on \( [0,1] \), say

\[ \alpha(x) = \lim_{k \to \infty} \alpha_{n_k}(x) \quad (0 \leq x \leq 1). \]

Lemma 3 guarantees that \( \alpha \in BV[0,1] \) and

\[ \lim_{k \to \infty} \int_0^1 f \, d\alpha_{n_k} = \int_0^1 f \, dx \quad (f \in C[0,1]). \quad (\text{iii}) \]

Using (i), (ii), and (iii) yields

\[ \Lambda(f) = \lim_{k \to \infty} \Lambda_0(B_{n_k}(f)) = \lim_{k \to \infty} \int_0^1 f \, d\alpha_{n_k} = \int_0^1 f \, dx \]

for all \( f \in C[0,1] \). Q.E.D.

Notes: If \( (X, \| \cdot \|) \) is a Banach space then the dual space of \( X \) is defined to be the vector space of all bounded
linear functionals $\Lambda : X \to \mathbb{R}$. The denote the dual space of $X$ by $X^*$. It is not hard to see that $X^*$ is a Banach space in its own right via the norm

$$\| \Lambda \| = \sup \{ |\Lambda(\tilde{x})| : \tilde{x} \in X, \| \tilde{x} \| = 1 \}.$$  

(cf. Proposition 10.3, Royden, p.221.) The Riesz Representation Theorem shows that $(C[a,b], \| \cdot \|_{\infty})^*$ can be identified with $BV[a,b]$.

Q: What is the Banach space norm $(\dagger)$ of the bounded linear functional $\Lambda_\alpha$ on $C[a,b]$ given by

$$\Lambda_\alpha(f) = \int_a^b f(x)dx,$$

where $\alpha \in BV[a,b]$?

A: $\| \Lambda_\alpha \| = \text{Var}(\alpha; a,b)$ which defines a norm on the subspace of $BV[a,b]$ consisting of those functions $\alpha \in BV[a,b]$ satisfying $\alpha(a)=0$.

Compare this norm with the Banach space norm

$$N_2(\alpha) = |\alpha(a)| + \text{Var}(\alpha; a,b)$$

on $BV[a,b]$. (See HW set #5, problem C.)