Mathematics 315 | Final Exam, Part I Spring 2011 | Name:______________________

This portion of the 200-point final examination is closed book/notes. You are to turn in your solutions to this portion before receiving the second part. It is recommended that you spend no longer than 70 minutes on this portion of the final exam.

1.(30 pts.) (a) State Littlewood's Three Principles.
(b) State a rigorous version for each one of Littlewood's principles.

2.(32 pts.) (a) State Lebesgue's Monotone Convergence Theorem.
(b) Give an example to show that the conclusion of the Monotone Convergence Theorem need not hold for a pointwise decreasing sequence of nonnegative measurable functions.
(c) State Fatou's Lemma.
(d) Give an example to show that the inequality in Fatou's Lemma can actually be strict.
(e) State Lebesgue's Dominated Convergence Theorem.
(f) Use Fatou's Lemma to prove the Dominated Convergence Theorem.

3.(33 pts.) In each of the following, compute the Lebesgue integral of f over the set E or show that f is not integrable over E. The symbol $\mathbb{Q}$ represents the set of rational numbers and P denotes the Cantor ternary set. Please justify the steps in your computations.

(a) $f(x)=\begin{cases} \sin(x) & \text{if } x\in P, \\ \dfrac{1}{\sqrt[3]{x}} & \text{if } x\in[0,1]\setminus P. \end{cases}$ $\qquad E=[0,1]$

(b) $f(x)=\begin{cases} x^2 & \text{if } x\in \mathbb{Q}, \\ e^{-|x|}\cos(x) & \text{if } x\in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$ $\qquad E=(0,\infty)$

(c) $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ $\qquad E=[0,\pi]$

Mathematics 315 Final Exam, Part II Name:____________________

Spring 2011

This portion of the 200 point final examination is "open book / open notes / open homework". That is, you may freely use your two textbooks for this class, Rudin's *Principles of Mathematical Analysis* and Royden's *Real Analysis*, your Math 315 class notes, and your solved homework problems from Math 315. **Work any THREE problems of your choosing, subject to the constraints that ONE problem must be chosen from Group A and TWO problems must be chosen from Group B.** Please **CIRCLE** the numbers of the problems on this portion whose solutions you wish me to grade.

Group A.

4.(35 pts.) Let f be the odd, 2π-periodic function determined by the formula

$$f(x) = x(\pi - x) \text{ if } 0 \le x \le \pi.$$

Show, by rigorous argument, that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^3} \sin\left[(2k+1)x\right] e^{-(2k+1)^2 t}$$

defines a function which solves the diffusion equation $u_t = u_{xx}$ in the region $t > 0$ of the xt-plane and which satisfies the initial condition $u(x,0) = f(x)$ for $-\infty < x < \infty$.

5.(35 pts.) (a) If $k \in \mathbb{Z}$ and $f(x) = e^{ikx}$, show that

(*) $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt.$$

(b) Show that (*) holds for every complex, continuous, 2π-periodic function f on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, 2π-periodic function f on $\mathbb{R}$? Prove your assertion.

Group B.

6.(35 pts.) Let E denote the set of real numbers in the interval $[0,1]$ which possess a decimal expansion which contains no 4's and no 7's. For instance, the numbers 1/2 = .5, and 7/10 = .6999... belong to E, while the numbers 4/9 = .4444... and $1/\sqrt{2} = .7071...$ do not.

(a) Compute the Lebesgue measure of E.

(b) Determine, with proof, whether E is a Borel set.

7.(35 pts.) Let $f \in L^1[0,1]$ and define $F(x) = \int_{[0,x]} f(t)\,dt$ for all $0 \le x \le 1$.

(a) Show that F is of bounded variation on the interval $[0,1]$.

(b) Show that F is a continuous function on the interval $[0,1]$.

(c) Find a simple formula in terms of the function f for the value of the Riemann-Stieltjes integral $\int_0^1 x\,dF$. If you have to use any result about Lebesgue integration whose proof we have not covered in Math 315 but which you suspect is true, please note this clearly at the point where it appears in your calculations.

8.(35 pts.) If $f \in L^1(\mathbb{R})$, define the Fourier transform of f at ξ by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

for all real numbers ξ.

(a) Show that $\hat{f}$ is a continuous function on the entire real line.

(b) If $\chi_{(a,b)}$ denotes the characteristic function of the bounded open interval (a,b), show that

$$\lim_{|\xi| \to \infty} \hat{\chi}_{(a,b)}(\xi) = 0.$$

(c) If E is a measurable subset of the real line with $m(E) < \infty$, show that

$$\lim_{|\xi| \to \infty} \hat{\chi}_E(\xi) = 0.$$

9.(35 pts.) Let $f_n : E \to [-\infty, \infty]$ $(n = 1, 2, 3, \ldots)$ be a sequence of Lebesgue integrable functions on E such that $f(x) = \lim_{n \to \infty} f_n(x)$ exists for almost every x in E and f is Lebesgue integrable on E. Show that $\int_E |f - f_n| dx \to 0$ if and only if $\int_E |f_n| dx \to \int_E |f| dx$.