Mathematics 315 | Final Exam, Part I Spring 2013 | Name:____________________

This portion of the 200 point final examination is "closed book". You are to turn in your solutions to the problems on this portion before receiving the second part. The recommended amount of time to spend on this portion of the exam is 60 minutes.

1.(30 pts.) (a) State Lebesgue's Monotone Convergence Theorem.

(b) Give an example to show that the conclusion of the Monotone Convergence Theorem need not hold for sequences $\langle f_n \rangle_{n=1}^{\infty}$ of measurable functions satisfying $f_n(x) \geq f_{n+1}(x) \geq 0$ for all x in E and all $n \geq 1$..

(c) State Fatou's Lemma.

(d) Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence in $L^1[0,1]$ such that $f_n(x) \to f(x)$ for all $x \in [0,1]$ and $f \in L^1[0,1]$. If $\int_{[0,1]} |f_n| dx \to \int_{[0,1]} |f| dx$, apply Fatou's Lemma to the sequence $h_n = |f_n| + |f| - |f_n - f|$ and show that

$$\int_{[0,1]} |f_n - f| dx \to 0.$$

(e) State Lebesgue's Dominated Convergence Theorem.

(f) Give an example to show that the equality in the conclusion of Lebesgue's Dominated Convergence Theorem may fail if there is no dominating integrable function g for the sequence $\langle f_n \rangle_{n=1}^{\infty}$.

2.(30 pts.) (a) State Littlewood's Three Principles.

(b) State a rigorous version for each one of Littlewood's principles.

3.(40 pts.) In each of the following, compute the Lebesgue integral of f over the set E or show that f is not integrable over E. The symbol $\mathbb{A}$ represents the set of algebraic numbers, $\mathbb{Q}$ stands for the set of rational numbers, and P denotes the Cantor ternary set. Please justify the steps in your computations.

(a) $$f(x) = \begin{cases} -1 & \text{if } x \in P, \\ 3 & \text{if } x \in [0,1] \setminus P, \\ 2 & \text{if } x \in [-1,0] \setminus \mathbb{Q}, \\ -4 & \text{if } x \in [-1,0] \cap \mathbb{Q}. \end{cases} \qquad E = [-1,1]$$

(b) $$f(x) = \begin{cases} \cos(x) & \text{if } x \in \mathbb{A}, \\ \dfrac{1}{x} & \text{if } x \in [0,1] \setminus \mathbb{A}. \end{cases} \qquad E = [0,1]$$

(c) $$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ e^{-|x|} \sin(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \qquad E = (-\infty, \infty)$$

(d) $$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \qquad E = [0, \pi]$$

Mathematics 315 | Final Exam, Part II Spring 2013 | Name:____________________ (1 pt.)

This portion of the 200 point final examination is "open book". That is, you may freely use your two textbooks for this class, Rudin's *Principles of Mathematical Analysis* and Royden's *Real Analysis*. **Work any THREE problems of your choosing, subject to the constraints that AT LEAST ONE problem must be chosen from EACH GROUP.** Please **CIRCLE** the numbers of the problems on this portion whose solutions you wish me to grade. All problems have the same point value, 33 points.

Group A.

1. (a) Determine, with proof, which of the following functionals define norms on the space BV[0,1] of functions of bounded variation on the interval [0,1].

$$N_1(f) = Var(f;0,1)$$
$$N_2(f) = |f(0)| + Var(f;0,1)$$
$$N_3(f) = \int_0^1 |f(x)|\,dx$$
$$N_4(f) = \sup\{|f(x)| : 0 \le x \le 1\}$$

(b) Determine with proof, which, if any, of the norms N above on BV[0,1] make the normed linear space (BV[0,1], N) a Banach space.

2. (a) State the Riesz Representation Theorem characterizing the bounded linear functionals on the Banach space $\left(C[0,1], \|\cdot\|_u\right)$.

(b) What form have all the bounded linear functionals on the Banach space $C^1[0,1]$ equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in [0,1]\} + \sup\{|f'(x)| : x \in [0,1]\}?$$

Justify your answer.

3. (a) If $k \in \mathbb{Z}$ and $f(x) = e^{ikx}$, show that

$$(*) \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

(b) Show that (*) holds for every complex, continuous, 2π – periodic function f on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, 2π – periodic function f on $\mathbb{R}$? Prove your assertion.

4. Let f be the 2π – periodic function defined on a fundamental period by the formulas $f(0) = 0$ and

$$f(x) = \frac{\pi - x}{2} \text{ if } 0 < x < 2\pi.$$

Show, by rigorous argument, that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-n^2 t}$$

defines a function which solves the diffusion equation $u_t = u_{xx}$ in the region $t > 0$ of the xt – plane and which satisfies the initial condition $u(x,0) = f(x)$ for $-\infty < x < \infty$.

Group B.

5. Let f be the function defined on the interval $[0,1]$ as follows: $f(x)=0$ if x is a point of the Cantor ternary set and $f(x)=1/k$ if x is in one of the complementary open intervals of the Cantor set with length 3^{-k}. For example, $f(1/3)=0$, $f(1/2)=1$, and $f(4/5)=1/2$.

(a) Show that f is a Lebesgue measurable function.

(b) Evaluate $\int_{[0,1]} f(x)dx$.

6. Let $f \in L^1(\mathbb{R})$ and define $\hat{f}$, the Fourier transform of f, at ξ by

$$\hat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

for all real numbers ξ.

(a) Show that $\hat{f}$ is a continuous function on the entire real line.

(b) If $\chi_{(a,b)}$ denotes the characteristic function of the bounded open interval (a,b), show that

$$\lim_{|\xi|\to\infty} \hat{\chi}_{(a,b)}(\xi)=0.$$

(c) If E is a measurable subset of the real line with $m(E)<\infty$, show that

$$\lim_{|\xi|\to\infty} \hat{\chi}_E(\xi)=0.$$

(d) Show that $\lim_{|\xi|\to\infty} \hat{f}(\xi)=0$.

In the following problems, let $\|f\|_{L^p}=\left(\int_{\mathbb{R}}|f(x)|^p dx\right)^{1/p}$ for $1\le p<\infty$ and $f\in L^p(\mathbb{R})$.

7. (a) Give an example of a sequence $\langle f_n\rangle$ of measurable functions on **R** with the following properties: $f_n(x)\to f(x)$ pointwise on **R**, $\|f_n\|_{L^1}\le M<\infty$ for all $n\ge 1$, and $\|f_n-f\|_{L^1}$ does not converge to 0 as $n\to\infty$.

(b) If $\{f_n\}$ is a sequence of measurable functions which converges to f pointwise on **R** and $\|f_n\|_{L^1}\to M<\infty$, what can you conclude about $\|f\|_{L^1}$? Justify this conclusion with a proof.

(c) If $\{f_n\}$ is a sequence of measurable functions which converges to f pointwise on **R** and $\|f_n\|_{L^1}\to\|f\|_{L^1}<\infty$, show that $\|f_n-f\|_{L^1}\to 0$.

8. (a) Let f be Lebesgue integrable on **R**. Show that

$$m(\{x\in\mathbf{R}:|f(x)|\ge\lambda\})\le\frac{\|f\|_{L^1}}{\lambda} \quad \text{for all } \lambda>0.$$

(b) Let f be a measurable function on **R** with the property that there is a positive number C such that

$$m(\{x\in\mathbf{R}:|f(x)|\ge\lambda\})\le\frac{C}{\lambda} \quad \text{for all } \lambda>0.$$

Is it true that $f\in L^1(\mathbf{R})$? Justify your answer.

(c) Generalize the results of (a) and (b) to $L^p(\mathbf{R})$ where $1<p<\infty$.