This is a three hour examination in which you may refer at any time to your textbooks for Math 315: Principles of Mathematical Analysis by Walter Rudin and Real Analysis by H.L. Royden. However, all other aids – books, lecture notes, homework and exam solutions, calculators, computers, smart phones, etc. – are NOT permitted.

This examination consists of six problems of equal value arranged in two groups. You are to solve FOUR problems of your choosing, subject to the constraint that **two problems must be chosen from Group A and two problems must be chosen from Group B**. The minimum score for a passing grade on this exam is 70 percent.
1. (a) If $k \in \mathbb{Z}$ and $f(x) = e^{ikx}$, show that

\[ (*) \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt. \]

(b) Show that (*) holds for every complex, continuous, $2\pi$ - periodic function $f$ on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, $2\pi$ - periodic function $f$ on $\mathbb{R}$? Prove your assertion.

2. Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... denote the (infinite) sequence of prime numbers, let $\pi(x)$ denote the number of primes less than or equal to $x$, and let $f(x) = \frac{1}{x}$ for $x > 0$.

(a) Show that

\[ \int_{1}^{x} f(t) \frac{dt}{\ln(t)} = \sum_{p \leq x} \frac{1}{p} \quad \text{for} \quad x > 2. \]

(b) Show that

\[ \int_{1}^{x} f(t) \frac{dt}{\ln(t)} = \sum_{p \leq x} \frac{\pi(t)}{t} + \int_{1}^{x} \frac{\pi(t)}{t^2} dt \quad \text{for} \quad x > 2. \]

(c) Use (a) and (b) to verify that

\[ \sum_{p \leq x} \frac{1}{p \ln(t)} = \ln(\ln(x)) + \int_{1}^{x} \left( \frac{\pi(t)}{t^2} \right) dt + \int_{1}^{x} \frac{\pi(t)}{t^2} dt + \frac{\pi(x)}{x} \quad \text{for} \quad x > 2. \]

In the remainder of this problem you may assume that to each $a > 0$ there correspond real constants $B = B(a) > 1$ and $C = C(a) > 0$ such that

\[ (*) \quad \left| \pi(x) - \frac{x}{\ln(x)} \right| \leq C x e^{-\alpha \sqrt{\ln(x)}} \quad \text{for} \quad x \geq B. \]

(d) Use (*) to help show that

\[ \lim_{x \to \infty} \left| \int_{1}^{x} \left( \frac{\pi(t)}{t^2} \right) dt \right| = 0. \]

(e) Why does the improper Riemann integral

\[ \int_{1}^{\infty} \left( \frac{\pi(t)}{t^2} \right) dt \]

converge?

(f) Use (c) and (*) to help show that

\[ \lim_{x \to \infty} \left( \sum_{p \leq x} \frac{1}{p} - \ln(\ln(x)) \right) = \int_{1}^{\infty} \left( \frac{\pi(t)}{t^2} \right) dt. \]

3. Let $f$ be the $2\pi$ - periodic function defined on a fundamental period by the formula

\[ f(x) = x^2 - \frac{\pi^2}{3} \quad \text{if} \quad -\pi \leq x < \pi. \]

Show, by rigorous argument, that

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)e^{-n^2t}. \]
defines a function which solves the diffusion equation \( u_t = u_{xx} \) in the region \( t > 0 \) of the \( xt \)-plane and which satisfies the initial condition \( u(x, 0) = f(x) \) for \(-\infty < x < \infty\).

**GROUP B**

4. Let \( f \) be a function defined and bounded on the unit square

\[ S = \{(x, t) : 0 < x < 1, \ 0 < t < 1\}. \]

Suppose that:

(a) for each fixed \( t \) in (0,1) the function \( x \mapsto f(x, t) \) is measurable,

(b) at each \( (x, t) \) in \( S \), the partial derivative \( \frac{\partial f}{\partial t} \) exists, and

(c) \( \frac{\partial f}{\partial t} \) is a bounded function in \( S \).

Show that \( \frac{d}{dt} \int_0^1 f(x, t) dx = \int_0^1 \frac{\partial f}{\partial t} (x, t) dx \).

5. Let \( \langle a_n \rangle_{n=1}^{\infty} \) be a positive divergent sequence, and for every positive integer \( n \) let

\[ f_n(x) = \begin{cases} a_n & \text{if } x \in \left( \frac{1}{n+1}, \frac{1}{n} \right), \\ 0 & \text{otherwise in } (0, 1). \end{cases} \]

(a) If \( \left\langle \frac{a_n}{n^2} \right\rangle_{n=1}^{\infty} \) is a bounded sequence, show that \( \left\langle \int_0^1 f_n dx \right\rangle_{n=1}^{\infty} \) is a bounded sequence.

(b) Place an X in each blank below that would imply

\[ \lim_{n \to \infty} \int_0^1 f_n dx = \int_0^1 \lim_{n \to \infty} f_n dx \]

and an O in each blank otherwise. Supply reasons for your answers.

(i) _________ \( \left\langle \frac{a_n}{\ln(n)} \right\rangle_{n=2}^{\infty} \) is a bounded sequence.

(ii) _________ \( \lim_{n \to \infty} \frac{a_n}{n^3 \ln\left(1 + \frac{1}{\sqrt{n}}\right)} = 0. \)

(iii) _________ \( \lim_{n \to \infty} \frac{a_n}{n^2} = 0. \)

(iv) _________ \( \left\langle \frac{a_n}{n^2 \ln(n)} \right\rangle_{n=2}^{\infty} \) is a bounded sequence.
6. Let \( f \) be a bounded measurable function on \([0,1]\) and define

\[
F(x) = \int_0^x f(t)\,dt \quad \text{for } x \in [0,1].
\]

(a) Show that \( F \) is continuous on \([0,1]\).
(b) Show that \( F \) is of bounded variation on \([0,1]\).

In the rest of this problem you may assume the following theorem: If \( g \) is increasing on \((a,b)\) then \( g'(x) \) exists a.e. in \((a,b)\).

(c) Why does \( F'(x) \) exist a.e. in \((0,1)\)?

(d) Use (c) to help show that \[
\int_0^y \{F'(t) - f(t)\} \, dt = 0 \quad \text{for all } y \text{ in } [0,1].
\]

(e) Use (c) and (d) to help show that \( F'(x) = f(x) \) a.e. in \([0,1]\).