Mathematics 315
Introduction to Mathematical Analysis
Qualifying Examination
August 2013

This is three hour examination in which you may refer at any time to your textbooks for Math 315: Principles of Mathematical Analysis by Walter Rudin and Real Analysis by H.L. Royden. However, all other aids – books, lecture notes, homework and exam solutions, calculators, computers, smart phones, etc. – are **NOT** permitted.

This examination consists of six problems of equal value arranged in two groups. You are to solve **FOUR** problems of your choosing, subject to the constraint that **two problems must be chosen from Group A and two problems must be chosen from Group B.** The minimum score for a passing grade on this exam is 70 percent.

GROUP A

1. (a) If $k \in \mathbb{Z}$ and $f(x) = e^{ikx}$, show that

(*) $$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

(b) Show that (*) holds for every complex, continuous, 2π – periodic function f on $\mathbb{R}$.

(c) Does (*) hold for every complex, bounded, measurable, 2π – periodic function f on $\mathbb{R}$? Prove your assertion.

2. Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ... denote the (infinite) sequence of prime numbers, let $\pi(x)$ denote the number of primes less than or equal to x, and let $f(x) = \frac{1}{x}$ for $x > 0$.

(a) Show that $\int_1^x f d\pi = \sum_{p_k \le x} \frac{1}{p_k}$ for $x > 2$.

(b) Show that $\int_1^x f d\pi = \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2} dt$ for $x > 2$.

(c) Use (a) and (b) to verify that

$$\sum_{p_k \le x} \frac{1}{p_k} = \ln(\ln(x)) + \int_e^x \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2} dt + \frac{\pi(x)}{x} \quad \text{for } x > 2.$$

In the remainder of this problem you may assume that to each $a > 0$ there correspond real constants $B = B(a) > 1$ and $C = C(a) > 0$ such that

(*) $$\left|\pi(x) - \frac{x}{\ln(x)}\right| \le Cxe^{-a\sqrt{\ln(x)}} \quad \text{for } x \ge B.$$

(d) Use (*) to help show that $\lim_{y > x \to \infty} \left|\int_x^y \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}\right| = 0.$

(e) Why does the improper Riemann integral $\int_e^\infty \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2}$ converge?

(f) Use (c) and (*) to help show that

$$\lim_{x\to\infty}\left(\sum_{p_k \le x} \frac{1}{p_k} - \ln(\ln(x))\right) = \int_e^\infty \left(\pi(t) - \frac{t}{\ln(t)}\right)\frac{dt}{t^2} + \int_1^e \frac{\pi(t)}{t^2} dt.$$

3. Let f be the 2π – periodic function defined on a fundamental period by the formula

$$f(x) = x^2 - \frac{\pi^2}{3} \quad \text{if } -\pi \le x < \pi.$$

Show, by rigorous argument, that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)e^{-n^2 t}$$

defines a function which solves the diffusion equation $u_t = u_{xx}$ in the region $t > 0$ of the xt-plane and which satisfies the initial condition $u(x,0) = f(x)$ for $-\infty < x < \infty$.

GROUP B

4. Let f be a function defined and bounded on the unit square

$$S = \{(x,t):\ 0 < x < 1,\ 0 < t < 1\}.$$

Suppose that:

(a) for each fixed t in $(0,1)$ the function $x \mapsto f(x,t)$ is measurable,

(b) at each (x,t) in S, the partial derivative $\dfrac{\partial f}{\partial t}$ exists, and

(c) $\dfrac{\partial f}{\partial t}$ is a bounded function in S.

Show that $\dfrac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \dfrac{\partial f}{\partial t}(x,t)dx.$

5. Let $\langle a_n \rangle_{n=1}^{\infty}$ be a positive divergent sequence, and for every positive integer n let

$$f_n(x) = \begin{cases} a_n & \text{if } x \in \left(\dfrac{1}{n+1}, \dfrac{1}{n}\right), \\ 0 & \text{otherwise in } (0,1). \end{cases}$$

(a) If $\left\langle \dfrac{a_n}{n^2} \right\rangle_{n=1}^{\infty}$ is a bounded sequence, show that $\left\langle \int_0^1 f_n dx \right\rangle_{n=1}^{\infty}$ is a bounded sequence.

(b) Place an X in each blank below that would imply

$$\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 \lim_{n\to\infty} f_n dx$$

and an O in each blank otherwise. Supply reasons for your answers.

(i) ________ $\left\langle \dfrac{a_n}{\ln(n)} \right\rangle_{n=2}^{\infty}$ is a bounded sequence.

(ii) ________ $\lim\limits_{n\to\infty} \dfrac{a_n}{n^3 \ln\left(1 + \dfrac{1}{\sqrt{n}}\right)} = 0.$

(iii) ________ $\lim\limits_{n\to\infty} \dfrac{a_n}{n^2} = 0.$

(iv) ________ $\left\langle \dfrac{a_n}{n^2 \ln(n)} \right\rangle_{n=2}^{\infty}$ is a bounded sequence.

6. Let f be a bounded measurable function on $[0,1]$ and define

$$F(x) = \int_0^x f(t)dt \quad \text{for } x \text{ in } [0,1].$$

(a) Show that F is continuous on $[0,1]$.
(b) Show that F is of bounded variation on $[0,1]$.

In the rest of this problem you may assume the following theorem: If g is increasing on (a,b) then $g'(x)$ exists a.e. in (a,b).

(c) Why does $F'(x)$ exist a.e. in $(0,1)$?

(d) Use (c) to help show that $\int_0^y \{F'(t) - f(t)\}\, dt = 0$ for all y in $[0,1]$.

(e) Use (c) and (d) to help show that $F'(x) = f(x)$ a.e. in $[0,1]$.