Mathematics 315
Introduction to Mathematical Analysis
Qualifying Examination for Mr. Hussam Abobaker
Spring 2013

This is a three hour examination in which you may refer at any time to your textbooks for Math 315: Principles of Mathematical Analysis by Walter Rudin and Real Analysis by H. L. Royden. However, no other aids (books, lecture notes, homework solutions, exam solutions, calculators, etc.) are permitted.

This examination consists of 8 problems of equal value, grouped into two parts. You are to solve **5 problems** of your choosing, subject to the constraint that **at least two problems must be chosen from Part I and at least two problems must be chosen from Part II**. The minimum score for a passing grade will be 70 percent.

PART I

1. Let $\alpha(x)$ denote the fractional part of the real number x. For example,

$$\alpha(5/4) = .25,\ \alpha(2) = 0, \text{ and } \alpha(\pi) = .1415926...$$

(a) Compute the total variation $T(\alpha;1,4)$ of α on the interval $[1,4]$.

(b) Show that the product of two functions of bounded variation on a closed bounded interval is of bounded variation on that interval.

(c) Let $f(x) = 1/x$ and $\beta(x) = \alpha^2(x)$. Why is f Riemann-Stieltjes integrable with respect to β on the interval $[1,4]$?

(d) Evaluate the Riemann-Stieltjes integral of f with respect to β on the interval $[1,4]$.

2. (a) Determine, with proof, which of the following functions define norms on the space BV[0,1] of functions of bounded variation on the interval [0,1].

$$N_1(f) = T(f;0,1)$$
$$N_2(f) = |f(0)| + T(f;0,1)$$
$$N_3(f) = \int_0^1 |f(x)|\,dx$$
$$N_4(f) = \sup\{|f(x)| : 0 \le x \le 1\}$$

(b) Determine with proof, which, if any, of the norms N above on BV[0,1] make the normed linear space (BV[0,1], N) a Banach space.

3. Let f be the odd, 2π-periodic function determined by $f(0) = 0$ and

$$f(x) = x(\pi - x) \quad \text{for} \quad 0 \le x \le \pi.$$

Show, by rigorous argument, that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8\sin\left((2k+1)x\right)e^{-(2k+1)^2 t}}{\pi(2k+1)^3}$$

defines a function which solves the diffusion equation $u_t - u_{xx} = 0$ in the upper halfplane $t > 0$ of the xt-plane, and which satisfies the initial condition $u(x,0) = f(x)$ for $-\infty < x < \infty$.

4. Let F be a continuous real function in the closed unit cube

$$Q = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

in $\mathbf{R}^3$. Show that to each positive number ε there corresponds a positive integer N and a (finite) collection f_k, g_k, h_k $(1 \le k \le N)$ of real polynomials in the interval [0,1] such that

$$\left| F(x,y,z) - \sum_{k=1}^{N} f_k(x)g_k(y)h_k(z) \right| < \varepsilon$$

for all (x,y,z) in Q.

PART II

5. Let f be the function defined on the interval $[0,1]$ as follows: $f(x)=0$ if x is a point of the Cantor ternary set and $f(x)=1/k$ if x is in one of the complementary open intervals of the Cantor set with length 3^{-k}. For example, $f(1/3)=0, f(1/2)=1,$ and $f(4/5)=1/2$.

(a) Show that f is a Lebesgue measurable function.

(b) Evaluate $\int_0^1 f(x)dx$.

6. Let $f \in L^1(\mathbb{R})$ and define the Fourier transform of f at ξ by

$$\hat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

for all real numbers ξ.

(a) Show that $\hat{f}$ is a continuous function on the entire real line.

(b) If $\chi_{(a,b)}$ denotes the characteristic function of the bounded open interval (a,b), show that

$$\lim_{|\xi|\to\infty} \hat{\chi}_{(a,b)}(\xi)=0.$$

(c) If E is a measurable subset of the real line with $m(E)<\infty$, show that

$$\lim_{|\xi|\to\infty} \hat{\chi}_E(\xi)=0.$$

(d) Show that $\lim_{|\xi|\to\infty} \hat{f}(\xi)=0$.

7. (a) Give an example of a sequence $\{f_n\}$ of measurable functions on **R** with the following properties: $f_n \to f$ pointwise on **R**, $\|f_n\|_{L^1} \le M < \infty$ for all $n \ge 1$, and $\|f_n - f\|_{L^1}$ does not converge to 0 as $n \to \infty$.

(b) If $\{f_n\}$ is a sequence of measurable functions which converges to f pointwise on **R** and $\|f_n\|_{L^1} \to M < \infty$, what can you conclude about $\|f\|_{L^1}$? Justify this conclusion with a proof.

(c) Show that if $\{f_n\}$ is a sequence of measurable functions which converges to f pointwise on **R** and $\|f_n\|_{L^1} \to \|f\|_{L^1} < \infty$, then $\|f_n - f\|_{L^1} \to 0$.

8. (a) Let f be Lebesgue integrable on **R**. Show that

$$m(\{x\in\mathbf{R}:|f(x)|\ge\lambda\}) \le \frac{\|f\|_{L^1}}{\lambda} \quad \text{for all } \lambda>0.$$

(b) Let f be a measurable function on **R** with the property that there is a positive number C such that

$$m(\{x\in\mathbf{R}:|f(x)|\ge\lambda\}) \le \frac{C}{\lambda} \quad \text{for all } \lambda>0.$$

Is it true that $f \in L^1(\mathbf{R})$? Justify your answer with either a proof or a counterexample.

(c) Generalize the results of (a) and (b) to $L^p(\mathbf{R})$ where $1<p<\infty$.