Solution to #13 on page 167 in Rudin.

#13. Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \le f_n(x) \le 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

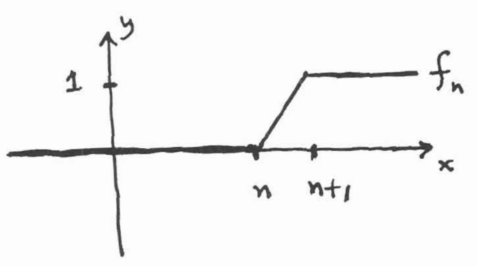
$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every x in $\mathbb{R}$. (The existence of such a pointwise convergent subsequence is usually called Helly's selection theorem.)

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on $\mathbb{R}$.

Note: The conclusion in part (b) is clearly false as the following sequence $\{f_n\}_{n=1}^{\infty}$ of monotonically increasing functions on $\mathbb{R}$, satisfying $0 \le f_n(x) \le 1$ for all x and all n, shows.

$$f_n(x) = \begin{cases} 0 & \text{if } x < n, \\ x-n & \text{if } n \le x \le n+1, \\ 1 & \text{if } n+1 < x. \end{cases}$$



It is clear that $f_n \to 0$ pointwise as $n \to \infty$ and observe that $f = 0$ is continuous on $\mathbb{R}$. However $|f_n(n+1) - f(n+1)| = 1$ for all $n \ge 1$ so no subsequence of $\{f_n\}_{n=1}^{\infty}$ can converge uniformly to f on $\mathbb{R}$.

Solution to #13(a): Let $E = \mathbb{Q} \cap [a,b]$. Theorem 7.23 implies the existence of a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\{f_{n_k}(x)\}_{k=1}^{\infty}$ converges for each x in E, say

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \qquad (x \in E).$$

It is clear that f is increasing on E. Define a function F on $[a,b]$ by

$$F(y) = \sup\{ f(x) : \; x \in E, \; x \le y \}.$$

It is easy to see that $F(y) = f(y)$ for all $y \in E$ and that F is increasing on $[a,b]$. We claim if $y \in (a,b) \setminus D(F)$ then $F(y) = \lim_{k\to\infty} f_{n_k}(y)$. To prove the claim, fix $y \in (a,b) \setminus D(F)$ and let $\varepsilon > 0$. Choose x and z in E with the following properties:

(i) $z < y < x$;

(ii) $f(z) > F(y) - \frac{\varepsilon}{2}$;

(iii) $f(x) < F(y) + \frac{\varepsilon}{2}$.

(Note that (ii) is possible by the definition of F as a supremum; (iii) is possible due to continuity of F at y, density of E in (a,b), and the fact that f and F agree on E.) Then choose positive integers $K_1 = K_1(z,\varepsilon)$ and $K_2 = K_2(x,\varepsilon)$ such that

(iv) $|f(z) - f_{n_k}(z)| < \frac{\varepsilon}{2}$ for all $k \ge K_1$, and

(v) $|f(x) - f_{n_k}(x)| < \frac{\varepsilon}{2}$ for all $k \ge K_2$.

If $k \ge \max\{K_1, K_2\}$ then (i) and $f_{n_k}\uparrow$ together with (ii) and (iv) imply

$$F(y) - f_{n_k}(y) \le F(y) - f_{n_k}(z) = F(y) - f(z) + f(z) - f_{n_k}(z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

while (i) and $f_{n_k}\uparrow$ together with (iii) and (v) yield

$$F(y) - f_{n_k}(y) \ge F(y) - f_{n_k}(x) = F(y) - f(x) + f(x) - f_{n_k}(x) > -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon .$$

That is, $|F(y) - f_{n_k}(y)| < \varepsilon$ for all $k \ge \max\{K_1, K_2\}$ and the claim is established.

Since F is increasing on $[a,b]$, $D(F)$ is at most countable by Theorem 4.30. Then Theorem 7.23 guarantees the existence of a subsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty}$ of $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{f_{n_{k_j}}(x)\}_{j=1}^{\infty}$ converges for all x in $\{a,b\} \cup D(F)$. Define

$$G(y) = \begin{cases} \lim\limits_{j\to\infty} f_{n_{k_j}}(y) & \text{if } y \in \{a,b\} \cup D(F), \\ F(y) & \text{if } y \in (a,b) \setminus D(F). \end{cases}$$

Clearly $f_{n_{k_j}} \longrightarrow G$ pointwise on $[a,b]$. Q.E.D.