On this exam, **no proofs are required** to support your answers if you are asked to state a theorem, write a formula, or give an example.

1. (34 pts.)
   (a) Define the phrase "E is a countable set".
   (b) Give an example of a subset E of the real numbers \( \mathbb{R} \) which is countable and dense in \( \mathbb{R} \).
   (c) Define the phrase "E is a subset of \( \mathbb{R} \) of measure zero".
   (d) Give an example of a subset E of \( \mathbb{R} \) which is countable and not of measure zero or state a theorem showing why this is impossible.
   (e) Give an example of a subset E of \( \mathbb{R} \) which is not countable and has measure zero or state a theorem showing why this is impossible.

   (f) If \( f \) is an increasing real function on \([a,b]\), state a theorem which characterizes the set of points at which \( f \) is continuous.

   (g) If \( f' \) is an increasing real function on \([a,b]\), state Lebesgue's theorem characterizing the set of points at which \( f \) is differentiable.

   Solve ONE of the following two problems, 2A or 2B. **CIRCLE** the number of the problem that you want me to grade.

2A. (33 pts.)
   Let \( f \) be a bounded real function on \([a,b]\) and let \( \alpha \) be an increasing real function on \([a,b]\).
   (a) Give a **careful and complete** definition of the phrase "\( f \) is Riemann-Stieltjes integrable with respect to \( \alpha \) on \([a,b]\)". (Note: Make sure that symbols such as \( U(P,f,\alpha) \), \( L(P,f,\alpha) \), \( \int_a^b f \, d\alpha \), and \( \int_a^b f \, d\alpha \) that appear in your definition are carefully defined.)
   (b) State a theorem which guarantees the existence of \( \int_a^b f \, d\alpha \).
   (c) If \( f \) is continuous on \([0,1]\) and \( \alpha(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} H(x - 1 + \frac{1}{2^n}) \), then write a formula for the value of \( \int_0^1 f \, d\alpha \). (Here \( H \) denotes the unit Heaviside step function.)

2B. (33 pts.)
   Let \( f \) be a bounded real function on \([a,b]\) and let \( \alpha \) be a function of bounded variation on \([a,b]\).
   (a) Define what it means for \( \alpha \) to be of bounded variation on \([a,b]\).
   (b) Give an example of a function which is differentiable but not of bounded variation on \([a,b]\).
   (c) State a condition on a differentiable function which will guarantee that it is of bounded variation on \([a,b]\).
   (d) State Jordan's theorem relating functions of bounded variation and increasing functions.
   (e) How is the Riemann-Stieltjes integral of \( f \) with respect to \( \alpha \) on \([a,b]\) defined in terms of Riemann-Stieltjes integrals with increasing integrators?
   (f) If \( f \) is Riemann integrable on \([0,1]\) and \( \alpha \) is differentiable with \( \alpha' \) Riemann integrable on \([0,1]\), then write a formula for the value of \( \int_0^1 f \, d\alpha \).
1. (a) $E$ is a countable set if there is a one-to-one function $f$ mapping the positive integers onto the set $E$.

(b) The rational numbers $\mathbb{Q}$ is an example of a subset of $\mathbb{R}$ which is countable and dense in $\mathbb{R}$.

(c) $E \subseteq \mathbb{R}$ is a set of measure zero if to each $\varepsilon > 0$ there corresponds a countable collection $\{(a_n, b_n)\}_{n=1}^{\infty}$ of open intervals in $\mathbb{R}$ satisfying $E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$.

(d) There is no such subset $E$ of $\mathbb{R}$ because if $E$ is countable then $E$ is of measure zero.

(e) The Cantor set $P$ in $[0,1]$ has measure zero and is not countable.

(f) If $f: [a,b] \to \mathbb{R}$ is increasing then the set of discontinuities of $f$ is either finite or countable.

(g) If $f: [a,b] \to \mathbb{R}$ is increasing then the set of points at which $f$ is not differentiable has measure zero.
2A. (a) Let \( P : a = x_0 < x_1 < x_2 < \ldots < x_n = b \) be a partition of \([a, b]\) and let \( M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \} \) and \( m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \} \) for \( i = 1, 2, \ldots, n \). Form the upper and lower Riemann–Stieltjes sums:

\[
\mathcal{U}(P, f, \alpha) = \sum_{i=1}^{n} M_i (\alpha(x_i) - \alpha(x_{i-1}))
\]

\[
\mathcal{L}(P, f, \alpha) = \sum_{i=1}^{n} m_i (\alpha(x_i) - \alpha(x_{i-1}))
\]

The upper and lower Riemann–Stieltjes integrals are:

\[
\int_{a}^{b} f \, d\alpha = \inf \{ \mathcal{U}(P, f, \alpha) : P \text{ is a partition of } [a, b] \}
\]

\[
\int_{a}^{b} f \, d\alpha = \sup \{ \mathcal{L}(P, f, \alpha) : P \text{ is a partition of } [a, b] \}
\]

We say that \( f \) is Riemann–Stieltjes integrable with respect to \( \alpha \) on \([a, b]\) provided

\[
\int_{a}^{b} f \, d\alpha = \int_{a}^{b} f \, d\alpha
\]

(b) If \( f \) is continuous on \([a, b]\) and \( \alpha \) is increasing on \([a, b]\) then \( f \in \mathcal{R}(\alpha) \) on \([a, b]\).

(or)

If \( f \) is monotonic on \([a, b]\) and \( \alpha \) is continuous and increasing on \([a, b]\) then \( f \in \mathcal{R}(\alpha) \) on \([a, b]\).

(c) If \( f \) is continuous on \([0, 1]\) and \( \alpha(x) = \sum_{n=1}^{\infty} 2^n H(x - 1 + 2^n) \) then

\[
\int_{0}^{1} f \, d\alpha = \sum_{n=1}^{\infty} 2^n f(1 - 2^{-n})
\]
The function \( f : [a, b] \rightarrow \mathbb{R} \) is of bounded variation on \([a, b]\) provided there is a real number \( M \) such that
\[
\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M
\]
for all partitions \( P: a = x_0 < x_1 < \ldots < x_n = b \) of \([a, b]\).

\( f(x) = \begin{cases} 
\frac{x^2 \sin(1/x^3)}{x^3} & \text{if } 0 < x \leq 1, \\
0 & \text{if } x = 0,
\end{cases} \)

is differentiable on \([0, 1]\) but \( f \) is not of bounded variation on \([0, 1]\).

If \( f \) is differentiable and \( f' \) is bounded on \([a, b]\), then \( f \) is of bounded variation on \([a, b]\).

(Jordan's Theorem)

If \( \alpha \) is of bounded variation on \([a, b]\) then there exist increasing real functions \( \alpha_1 \) and \( \alpha_2 \) on \([a, b]\) such that \( \alpha(x) = \alpha_1(x) - \alpha_2(x) \) for all \( a \leq x \leq b \).

Let \( \alpha = \alpha_1 - \alpha_2 \) where each \( \alpha_i \) is increasing on \([a, b]\). If \( f \in R(\alpha_i) \) on \([a, b]\) for \( i = 1, 2 \), then define \( \int_a^b f \, d\alpha = \int_a^b f \, d\alpha_1 - \int_a^b f \, d\alpha_2 \).

If \( f \in R([0, 1]) \) and \( \alpha \) is differentiable with \( \alpha' \in R([0, 1]) \), then \( f \in R(\alpha) \) and
\[
\int_0^1 f \, d\alpha = \int_0^1 f(x) \alpha'(x) \, dx.
\]
3A. (a) $N$ is a norm on the vector space $\mathbb{X}$ provided $N$ is a real function defined on $\mathbb{X}$ with the following properties:

(i) $N(\vec{x}) \geq 0$ for all $\vec{x}$ in $\mathbb{X}$, with equality only if $\vec{x} = 0$;
(ii) $N(c\vec{x}) = |c|N(\vec{x})$ for all $\vec{x}$ in $\mathbb{X}$ and $c$ in $\mathbb{R}$;
(iii) $N(\vec{x} + \vec{y}) \leq N(\vec{x}) + N(\vec{y})$ for all $\vec{x}$ and $\vec{y}$ in $\mathbb{X}$.

(b) A sequence $\{\vec{x}_n\}_{n=1}^{\infty}$ in a normed linear space $(\mathbb{X}, N)$ is convergent provided there exists $\vec{x}$ in $\mathbb{X}$ such that $\lim_{n \to \infty} N(\vec{x}_n - \vec{x}) = 0$.

(c) A sequence $\{\vec{x}_n\}_{n=1}^{\infty}$ in a normed linear space $(\mathbb{X}, N)$ is Cauchy provided $N(\vec{x}_n - \vec{x}_m) \to 0$ as $m$ and $n$ tend to infinity.

(d) Convergent sequences are Cauchy sequences in a normed linear space. However, for a general normed linear space, Cauchy sequences need not converge.

(e) A normed linear space $(\mathbb{X}, N)$ in which every Cauchy sequence is convergent is called a Banach space.

(f) $(C[0,1], \| \cdot \|_1)$, where $\| f \|_1 = \int_0^1 |f(x)| \, dx$ for $f$ in $C[0,1]$, is a normed linear space which is not a Banach space.

(g) $(C[0,1], \| \cdot \|_\infty)$, where $\| f \|_\infty = \sup\{ |f(x)| : 0 \leq x \leq 1 \}$ for $f$ in $C[0,1]$, is a Banach space.
\(3B. \) (a) \( \{f_n\}_{n=1}^\infty \) converges to \( f \) pointwise on \([a, b]\) provided
\[
\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for each} \quad x \in [a, b].
\]
(b) \( \{f_n\}_{n=1}^\infty \) converges to \( f \) uniformly on \([a, b]\) provided to each \( \varepsilon > 0 \) there corresponds an integer \( N = N(\varepsilon) \geq 1 \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \in [a, b] \) and all \( n \geq N \).

(c) Let \( f_n(x) = \begin{cases} 1 - 2n|x - \frac{1}{n}| & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x \leq 1, \end{cases} \)
and \( f(x) = 0 \) if \( 0 \leq x \leq 1 \). Show \( \{f_n\}_{n=1}^\infty \) converges pointwise to \( f \) on \([0, 1]\) but \( \{f_n\}_{n=1}^\infty \) does not converge uniformly to \( f \) on \([0, 1]\).

(d) Let \( f_n : E \to \mathbb{R} \quad (n=1, 2, 3, \ldots) \) be a sequence of bounded functions on a set \( E \) and let \( M_n = \sup \{|f_n(x)| : x \in E\} \) \((n=1, 2, 3, \ldots)\). If \( \sum_{n=1}^\infty M_n < \infty \)
then the sequence of partial sums of \( \sum_{n=1}^\infty f_n \) converges uniformly on \( E \).

(e) Let \( \mathcal{A} \) be a family of real functions defined on a set \( E \). \( \mathcal{A} \) is called an algebra provided:
(i) if \( f \) and \( g \) belong to \( \mathcal{A} \) then \( f+g \) belongs to \( \mathcal{A} \);
(ii) if \( f \) belongs to \( \mathcal{A} \) and \( c \) is any real number then \( cf \) belongs to \( \mathcal{A} \);
(iii) if \( f \) and \( g \) belong to \( \mathcal{A} \) then \( fg \) belongs to \( \mathcal{A} \).

(f) \( \mathcal{A} \) separates points on \( E \) if to each pair of distinct \( p \) and \( q \) in \( E \) there corresponds \( f \) in \( \mathcal{A} \) such that \( f(p) \neq f(q) \).
(g) A vanishes at no point of $E$ if to each point $p$ in $E$ there corresponds $f$ in $A$ such that $f(p) \neq 0$.

(h) Let $A$ be an algebra of real continuous functions on a compact metric space $K$. If $A$ separates points on $K$ and if $A$ vanishes at no point of $K$ then to each continuous function $f : K \to \mathbb{R}$ there corresponds a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $A$ such that $f_n \to f$ uniformly on $K$. 
Math 315  
Midterm Exam  
Spring 2011  

\( n: 14 \)  
mean: 77.1  
standard deviation: 15.2

**Distribution of Scores:**  

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