The last two pages of this exam consist of a table of Fourier transforms and some convergence theorems for Fourier series. Furthermore, you may find the following formulas useful on this exam.

\[ \int_{0}^{\pi} \frac{dz}{a + b \cos(z)} = \frac{2}{\sqrt{a^2 - b^2}} \arctan \left( \frac{a - b}{a + b} \tan \left( \frac{x}{2} \right) \right) \quad \text{if} \quad a^2 > b^2 \quad \text{and} \quad -\pi < x < \pi \]

\[ \nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) \frac{\partial u}{\partial \phi} \right) + \frac{1}{r^2 \sin^2(\phi)} \frac{\partial^2 u}{\partial \theta^2} \]

1.(33 pts.) Classify the partial differential equation \( u_{xx} + 2u_{xy} - 3u_{yy} + u_x + 2u_y = 0 \) as hyperbolic, elliptic, parabolic, or none of these. If it is possible, find the general solution of this partial differential equation in the \( xy \)-plane. If it is not possible, please explain why this is so.

2.(34 pts.) (a) Use Fourier transform methods to derive the formula

\[ u(x,t) = \frac{1}{2} \int_{0}^{\pi} f(s, \tau) ds \]

for a solution to the inhomogeneous wave equation in the \( xt \)-plane,

\[ u_{tt} - u_{xx} = f(x, t) \]

satisfying homogeneous initial conditions: \( u_x(x, 0) = 0 = u_t(x, 0) \) for \( -\infty < x < \infty \)

(b) Compute a solution to \( u_{tt} - u_{xx} = xt \) in the \( xt \)-plane which satisfies homogeneous initial conditions.

(c) Is the solution to the problem in part (b) unique? Justify your answer.

3.(33 pts.) Solve \( u_t - u_{xx} = 0 \) in the upper half-plane \( -\infty < x < \infty \), \( 0 < t < \infty \), subject to the initial condition \( u(x, 0) = e^{-x^2} \) for \( -\infty < x < \infty \).

4.(34 pts.) The material in a thin circular disk of unit radius has reached a steady-state temperature distribution. The material is held at 100 degrees Centigrade on the top half of the disk’s edge and at 0 degrees Centigrade on the bottom half of its edge.

(a) Write the partial differential equation and boundary condition(s) governing the steady-state temperature function of the material.

(b) Write a formula for the steady-state temperature function of the material.

(c) What is the steady-state temperature of the material at the center of the disk? Justify your answer.

(d) What is the steady-state temperature of the material at a general point in the disk? (You must evaluate any integral expressions for credit on this part of the problem.)

5.(33 pts.) (a) Show that the Fourier sine series \( \sum_{n=1}^{\infty} b_n \sin(n \pi \theta) \) of \( f(\theta) = 4\theta(1-\theta) \) on \([0,1]\) is

\[ 4\theta(1-\theta) \approx \frac{32}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi \theta)}{(2k-1)^3} \]

(b) Show that the Fourier sine series of \( f \) converges uniformly to \( f \) on the interval \([0,1]\).
(c) Use the results of the previous parts of this problem to find the sum of the series \[ \sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}. \]

6. (33 pts.) Solve \( \nabla^2 u = 0 \) in the unit cube \( 0 < x < 1, \ 0 < y < 1, \ 0 < z < 1, \) given that
\[ u(x,y,1) = 16xy(1-x)(1-y) \text{ for } 0 \leq x \leq 1, \ 0 \leq y \leq 1, \]
and that \( u \) satisfies homogeneous Dirichlet boundary conditions on the other five faces of the cube. (You may find the results of problem 5 useful.)
A Brief Table of Fourier Transforms

<table>
<thead>
<tr>
<th>A.</th>
<th>$f(x)$ if $-b &lt; x &lt; b$, 0 otherwise.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.</td>
<td>$f(x)$ if $c &lt; x &lt; d$, 0 otherwise.</td>
</tr>
<tr>
<td>C.</td>
<td>$\frac{1}{x^2 + a^2}$ ((a &gt; 0))</td>
</tr>
<tr>
<td>D.</td>
<td>$x$ if $0 &lt; x \leq b$, 2$b-x$ if $b &lt; x &lt; 2b$, 0 otherwise.</td>
</tr>
<tr>
<td>E.</td>
<td>$e^{-ax}$ if $x &gt; 0$, 0 otherwise.</td>
</tr>
<tr>
<td>F.</td>
<td>$e^{ax}$ if $b &lt; x &lt; c$, 0 otherwise.</td>
</tr>
<tr>
<td>G.</td>
<td>$e^{-ax}$ if $-b &lt; x &lt; b$, 0 otherwise.</td>
</tr>
<tr>
<td>H.</td>
<td>$e^{ax}$ if $c &lt; x &lt; d$, 0 otherwise.</td>
</tr>
<tr>
<td>I.</td>
<td>$e^{-ax^2}$ ((a &gt; 0))</td>
</tr>
<tr>
<td>J.</td>
<td>$\frac{\sin(\alpha x)}{x}$ ((a &gt; 0))</td>
</tr>
</tbody>
</table>

\[\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx\]

\[
\begin{align*}
&\frac{\sqrt{2}}{\pi} \frac{\sin(b\xi)}{\xi} \\
&\frac{\sqrt{2}}{\pi} \frac{e^{-ib\xi} - e^{-id\xi}}{\xi^2} \\
&\frac{1}{\sqrt{2}a} e^{-\xi^2/(4a)} \left\{ \begin{array}{ll}
0 & \text{if } |\xi| \geq a, \\
\sqrt{\frac{\pi}{2}} & \text{if } |\xi| < a.
\end{array} \right.
\end{align*}
\]
Convergence Theorems

Consider the eigenvalue problem

\[ X''(x) + \lambda X(x) = 0 \quad \text{in} \quad a < x < b \quad \text{with any symmetric boundary conditions} \]

and let \( \Phi = \{X_1, X_2, X_3, \ldots\} \) be the complete orthogonal set of eigenfunctions for (1). Let \( f \) be any absolutely integrable function defined on \( a \leq x \leq b \). Consider the Fourier series for \( f \) with respect to \( \Phi \):

\[
f(x) \approx \sum_{n=1}^{\infty} A_n X_n(x)
\]

where

\[
A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \quad (n = 1, 2, 3, \ldots).
\]

**Theorem 2.** (Uniform Convergence) If

(i) \( f(x), f'(x), \) and \( f''(x) \) exist and are continuous for \( a \leq x \leq b \) and

(ii) \( f \) satisfies the given symmetric boundary conditions,

then the Fourier series of \( f \) converges uniformly to \( f \) on \([a, b]\).

**Theorem 3.** (\( L^2 \)-Convergence) If

\[
\int_a^b |f(x)|^2 \, dx < \infty
\]

then the Fourier series of \( f \) converges to \( f \) in the mean-square sense in \((a, b)\).

**Theorem 4.** (Pointwise Convergence of Classical Fourier Series)

(i) If \( f \) is a continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) at \( x \) converges pointwise to \( f(x) \) in the open interval \( a < x < b \).

(ii) If \( f \) is a piecewise continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) converges pointwise at every point \( x \) in \((-\infty, \infty)\). The sum of the Fourier series is

\[
\sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2}
\]

for all \( x \) in the open interval \((a, b)\).

**Theorem 4∞.** If \( f \) is a function of period \( 2l \) on the real line for which \( f \) and \( f' \) are piecewise continuous, then the classical full Fourier series converges to \( \frac{f(x^+) + f(x^-)}{2} \) for every real \( x \).
\#1. \[ u_{xx} - 3u_{xy} + 2u_{yy} + u_x + 2uy = 0 \] \((*)\)

\[ B^2 - 4AC = (-3)^2 - 4(1)(2) = 1 > 0. \]

Therefore the PDE \((*)\) is hyperbolic. In order to solve the PDE we factor the differential operator associated with the second order terms in \((*)\):

\[ \frac{\partial^2}{\partial x^2} - 3 \frac{\partial^2}{\partial x \partial y} + 2 \frac{\partial^2}{\partial y^2} = \left( \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right). \]

Therefore \((*)\) is equivalent to

\[ \left( \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) u + \left( \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y} \right) u = 0. \]

The suggested substitution in this case is

\[ \xi = 2x + y, \]
\[ \eta = x + y. \]

Then the chain rule implies

\[ \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = 2 \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \]
\[ \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \]

so

\[ \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} = 2 \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 2 \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) = -\frac{\partial}{\partial \eta}, \]
\[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) = \frac{\partial}{\partial \xi}, \]
\[ \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + 2 \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) = 4 \frac{\partial}{\partial \xi} + 3 \frac{\partial}{\partial \eta}. \]

Consequently \((*)\) is equivalent to

\[ -\frac{\partial u}{\partial \eta} + 4 \frac{\partial u}{\partial \xi} + 3 \frac{\partial u}{\partial \eta} = 0. \]
Unfortunately, the techniques that allow us to solve \( \frac{\partial^2 u}{\partial x^2} + \alpha u = 0 \) or \( \frac{\partial^2 u}{\partial x^2} + \beta u \)
do not apply to (**) Therefore we conclude that it is not possible
to solve (*)&), at least with the techniques of solution covered in
this course.

**Note 1:** Using the method of separation of variables in (**) leads to
formal solutions of the form

\[
u(x, \eta) = \sum_{n=1}^{\infty} c_n e^{(\lambda_n x + \lambda_n \eta)}
\]

where \( c_1, c_2, \ldots \) and \( \lambda_1, \lambda_2, \ldots \) are arbitrary constants.

**Note 2:** Using the change of dependent variable \( u(x, y) = v(x, y) e^{10x + 7y} \)
in (**) produces the equivalent equation

\[
(+) \quad \left( \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)v + 12v = 0.
\]

Then the change of independent variables

\[
x = z
\]

\[
\eta = 3x + 2y
\]
in (**) leads to the equivalent equation

\[
(\dagger) \quad \frac{\partial^2 v}{\partial z^2} - \frac{\partial^2 v}{\partial \eta^2} + 12v = 0.
\]

But (\dagger) is a special case of the telegraph equation

\[
\frac{\partial^2 v}{\partial z^2} - \frac{c^2 \partial^2 v}{\partial \eta^2} + 2\beta \frac{\partial v}{\partial z} + \alpha v = 0
\]
with \( c = 1, \beta = 0, \) and \( \alpha = 12. \) (See [PDEs with BVPs with Applications](https://example.com) (third edition) by Mark Pinsky.) It can be shown that the solution to (44) satisfying the initial conditions

\[
(44) \quad \nu(\eta, 0) = f_1(\eta) \quad \text{and} \quad \frac{\partial \nu}{\partial \xi}(\eta, 0) = f_2(\eta) \quad \text{for} \quad -\infty < \eta < \infty
\]

is

\[
\nu(\eta, \xi) = \frac{1}{2} \int_{-3}^{3} f_2(\eta + s) J_0\left(\sqrt{12\frac{s^2}{3} - 12\xi^2}\right) ds
\]

\[
+ \frac{1}{2} \left( f_1(\eta + 3) + f_1(\eta - 3) \right)
\]

\[
- \frac{\sqrt{12}}{2} \int_{-3}^{3} f_2(\eta + s) J_1\left(\sqrt{12\frac{s^2}{3} - 12\xi^2}\right) ds
\]

where \( J_0 \) and \( J_1 \) are the Bessel functions of orders 0 and 1, respectively. (See Pinsky’s book cited above, pp. 461–462 and exercise 8.5.4.) Of course, this is far beyond what was covered in our Math 325 class but I thought people might want to know how to solve this PDE.
(b) See the bonus portion of problem 3 on Exam II where it is shown that \( u(x,t) = \frac{x^3}{6} \).

(c) Suppose that \( u \) is a solution to the problem in part (a) such that, for each fixed \( t \) in \((-\infty, \infty)\), \( u, u_t, u_x, u_{xt}, u_{xx}, \) and \( u_{tt} \) are square-integrable functions of \( x \) and \( \lim_{|x| \to \infty} u(x,t) = 0 = \lim_{|x| \to \infty} u_t(x,t) \). Then \( u \) is unique.

Reason: Let \( v = v(x,t) \) be another such solution to the problem in part (a) and define \( w(x,t) = u(x,t) - v(x,t) \) for all real \( x \) and \( t \). Then \( w \) solves \( w_{tt} - w_{xx} = 0 \) in the \( xt \)-plane and \( w(x,0) = 0 = w_t(x,0) \) for \(-\infty < x < \infty\).

Consider the energy function of \( w \):

\[
E(t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t) \right] dx \quad \text{if} \quad -\infty < t < \infty.
\]

[Note that the square-integrability hypotheses imply that the energy function of \( w \) is finite for all \( t \) in \((-\infty, \infty)\).] The square-integrability hypotheses allow differentiation of the energy function of \( w \):

\[
\frac{dE}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \left[ \frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t) \right] dx = \int_{-\infty}^{\infty} \left[ w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx.
\]

and the decay conditions and integration by parts show

\[
\frac{dE}{dt} = \int_{-\infty}^{\infty} w_t(x,t) w_{tt}(x,t) dx + w_t(x,t) w_{x}(x,t) \bigg|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} w_t(x,t) w_{xx}(x,t) dx
\]

\[
= \int_{-\infty}^{\infty} w_t(x,t) \left[ w_{tt}(x,t) - w_{xx}(x,t) \right] dx = 0,
\]

so \( E = E(t) \) is a constant function on \((-\infty, \infty)\). Since
\[ E(t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} w_t^2(x,0) + \frac{1}{2} w_x^2(x,0) \right] dx = 0 \]

It follows that \( E(t) = 0 \) for all \(-\infty < t < \infty\). By the vanishing theorem, \( w_t(x,t) = 0 = w_x(x,t) \) for all \(-\infty < x < \infty\) and each fixed \( t \) in \((-\infty, \infty)\). Then \( w(x,t) = \text{constant in the } xt\)-plane and \( w(x,0) = 0 \) implies \( w(x,t) = 0 \) for all \(-\infty < x < \infty\) and all \(-\infty < t < \infty\). That is, \( v \equiv u \) so the solution to the problem in part (a) satisfying the square-integrability and decay conditions is unique.

### 3

See problem 1 on Exam III where it is shown that

\[
 u(x,t) = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}.
\]

### 4 (a)

Let \( u = u(r, \theta) \) denote the steady-state temperature of the material in the disk at the point whose polar coordinates are \((r, \theta)\). Then \( u \) satisfies

\[
\nabla^2 u = 0 \quad \text{in} \quad 0 < r < 1, \quad -\pi \leq \theta \leq \pi,
\]

\[
u(1; \theta) = h(\theta) = \begin{cases} 100 & \text{if} \quad 0 < \theta < \pi, \\ 0 & \text{if} \quad -\pi < \theta < 0. \end{cases}
\]
(b) Poisson's formula

\[ u(r; \theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{h(\phi) d\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2} \]

applied to the problem in part (a) gives

\[ u(r; \theta) = \frac{1 - r^2}{2\pi} \int_{0}^{\pi} 100 \, d\phi \quad (0 \leq r < 1, -\pi \leq \theta \leq \pi) \]

(c) We substitute \( r = 0 \) in the formula in part (b) (or apply the mean value theorem for harmonic functions) to obtain the steady-state temperature of the material at the center of the disk:

\[ u(0; \theta) = \frac{1}{2\pi} \int_{0}^{\pi} 100 \, d\phi = 50 \]

(d) Let \( z = \phi - \theta \) in the formula for \( u \) in part (b). Then

\[ u(r; \theta) = \frac{1 - r^2}{2\pi} \int_{-\theta}^{\pi-\theta} \frac{100 \, dz}{1 + r^2 - 2r \cos(z)} \]

\[ = \frac{50(1 - r^2)}{\pi} \left( \int_{0}^{\pi-\theta} \frac{dz}{1 + r^2 - 2r \cos(z)} - \int_{0}^{-\theta} \frac{dz}{1 + r^2 - 2r \cos(z)} \right) \]

Suppose that \( 0 < \theta < \pi \). Then \( 0 < \pi - \theta < \pi \) and \( -\pi < -\theta < 0 \) so by the integration formula on the first page of the exam with \( a = 1 + r^2 \) and \( b = -2r \), we have

\[ u(r; \theta) = \frac{50(1 - r^2)}{\pi} \cdot \frac{2}{\sqrt{(1 + r^2)^2 - (2r)^2}} \left[ \arctan\left( \frac{1 + r^2 + 2r \tan(\pi - \theta)}{1 + r^2 - 2r} \right) - \arctan\left( \frac{1 + r^2 - 2r \tan(\theta)}{1 + r^2 - 2r} \right) \right] \]
Simplifying yields

\[(***) \quad u(r; \theta) = \frac{100}{\pi} \left[ \text{Arctan} \left( \frac{1+r}{1-r} \cot \left( \theta/2 \right) \right) + \text{Arctan} \left( \frac{1+r}{1-r} \tan \left( \theta/2 \right) \right) \right] \quad \text{if} \quad 0 < \theta < \pi.
\]

Suppose \(-\pi < \theta < 0\). Then \(0 < -\theta < \pi\) and \(\pi < \pi - \theta < 2\pi\) so

\[(***) \quad - \int_0^{\pi - \theta} \frac{dz}{1 + r^2 - 2r \cos z} = \frac{2}{1-r^2} \text{Arctan} \left( \frac{1+r}{1-r} \tan \left( \theta/2 \right) \right)
\]

as before, and

\[(***) \quad \int_0^{\pi - \theta} \frac{dz}{1 + r^2 - 2r \cos z} = \int_0^{\pi} \frac{dz}{1 + r^2 - 2r \cos z} + \int_{\pi}^{\pi - \theta} \frac{dz}{1 + r^2 - 2r \cos z}
\]

Let \(w = z - \pi\)

\[= \frac{2}{1-r^2} \cdot \frac{\pi}{2} + \int_0^{-\theta} \frac{dw}{1 + r^2 + 2r \cos w}
\]

\[= \frac{2}{1-r^2} \cdot \frac{\pi}{2} + \frac{2}{1-r^2} \cdot \text{Arctan} \left( \frac{1-r}{1+r} \tan \left( \theta/2 \right) \right)
\]

\[= \frac{2}{1-r^2} \cdot \frac{\pi}{2} - \frac{2}{1-r^2} \cdot \text{Arctan} \left( \frac{1-r}{1+r} \tan \left( \theta/2 \right) \right).
\]

Using (**), (***) and (***) we have

\[u(r; \theta) = \frac{100}{\pi} \left[ \frac{\pi}{2} - \text{Arctan} \left( \frac{1-r}{1+r} \tan \left( \theta/2 \right) \right) + \text{Arctan} \left( \frac{1-r}{1+r} \tan \left( \theta/2 \right) \right) \right]
\]

provided \(-\pi < \theta < 0\). Collecting (**) and (***) produces the solution to the steady-state temperature problem in part(a):

\[u(r; \theta) = \begin{cases} \frac{100}{\pi} \left[ \text{Arctan} \left( \frac{1+r}{1-r} \cot \left( \theta/2 \right) \right) + \text{Arctan} \left( \frac{1+r}{1-r} \tan \left( \theta/2 \right) \right) \right] & \text{if} \quad 0 < \theta < \pi \\ \frac{100}{\pi} \left[ \frac{\pi}{2} - \text{Arctan} \left( \frac{1-r}{1+r} \tan \left( \theta/2 \right) \right) + \text{Arctan} \left( \frac{1-r}{1+r} \tan \left( \theta/2 \right) \right) \right] & \text{if} \quad -\pi < \theta < 0 \end{cases}
\]
Note: Using the identity \( \tan(x + \beta) = \frac{\tan(x) + \tan(\beta)}{1 - \tan(x)\tan(\beta)} \), one can show that the solution to the problem in part (a) of problem 4 is

\[
W(r; \theta) = 50 + \frac{100}{\pi} \arctan \left( \frac{2 \rho \sin \theta}{1 - r^2} \right)
\]

\#5

(a) See part (a) of problem 1 on Exam III.

(b) See part (c) of problem 1 on Exam III.

(c) See part (e) of problem 1 on Exam III where it was shown that

\[
\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} = \frac{\pi^6}{960}
\]

\#6

We seek the solution of the boundary value problem:

1. \( u_{xx} + u_{yy} + u_{zz} = 0 \) if \( 0 < x < 1, 0 < y < 1, 0 < z < 1 \),

2-3. \( u(x, 0, z) = 0 = u(x, 1, z) \) if \( 0 \leq x \leq 1, 0 \leq z \leq 1 \),

4-5. \( u(0, y, z) = 0 = u(1, y, z) \) if \( 0 \leq y \leq 1, 0 \leq z \leq 1 \),

6-7. \( u(x, y, 0) = 0 \) and \( u(x, y, 1) = 16xy(1-x)(1-y) \) if \( 0 \leq x \leq 1, 0 \leq y \leq 1 \)
We use the method of separation of variables. We seek nontrivial solutions to the homogeneous part of the problem, \( \text{\textcircled{1}} - \text{\textcircled{2}} - \text{\textcircled{3}} - \text{\textcircled{4}} - \text{\textcircled{5}} - \text{\textcircled{6}} \), of the form
\[
\begin{align*}
    u(x, y, z) &= \Xi(x) \Upsilon(y) \Psi(z) .
\end{align*}
\]
Substituting this expression for \( u \) in \( \text{\textcircled{1}} \) and simplifying yields
\[
\begin{align*}
    \frac{\Xi''(x)}{\Xi(x)} + \frac{\Upsilon''(y)}{\Upsilon(y)} &= -\frac{\Psi''(z)}{\Psi(z)} = \text{constant} = \lambda \quad \text{and}
\end{align*}
\]
then
\[
\begin{align*}
    \frac{\Xi''(x)}{\Xi(x)} - \lambda = -\frac{\Upsilon''(y)}{\Upsilon(y)} = \text{constant} = \mu .
\end{align*}
\]
Substituting \( u(x, y, z) = \Xi(x) \Upsilon(y) \Psi(z) \) in \( \text{\textcircled{4}} - \text{\textcircled{5}} \) and using the fact that \( u \) is not identically zero produces \( \Xi(0) = 0 = \Xi(1) \).

Similarly, \( \text{\textcircled{2}} - \text{\textcircled{3}} \) leads to \( \Upsilon(0) = 0 = \Upsilon(1) \) and \( \text{\textcircled{6}} \) leads to \( \Psi(0) = 0 \). Therefore, we are led to the system
\[
\begin{align*}
    \Xi''(x) + \lambda \Xi(x) &= 0 & \text{if} \quad 0 < x < 1, & \Xi(0) = 0 = \Xi(1), \\
    \Upsilon''(y) + \mu \Upsilon(y) &= 0 & \text{if} \quad 0 < y < 1, & \Upsilon(0) = 0 = \Upsilon(1), \\
    \Psi''(z) - (\lambda + \mu) \Psi(z) &= 0 & \text{if} \quad 0 < z < 1, & \Psi(0) = 0 .
\end{align*}
\]

The solutions to \( \Xi \) and \( \Upsilon \) eigenvalue problems with Dirichlet boundary conditions are, respectively,
\[
\begin{align*}
    \lambda_\ell &= (\ell \pi)^2 , \quad \Xi_\ell (x) &= \sin(\ell \pi x) & (\ell = 1, 2, 3, \ldots) , \\
    \mu_m &= (m \pi)^2 , \quad \Upsilon_m (y) &= \sin(m \pi y) & (m = 1, 2, 3, \ldots) .
\end{align*}
\]

The solution, up to a constant multiple, of the corresponding \( \Psi \)-problem when \( \lambda = \lambda_\ell \) and \( \mu = \mu_m \) is
\[
\Xi_\ell (x) = \sinh(\pi z \sqrt{\ell^2 + m^2}) & (\ell = 1, 2, 3, \ldots ; m = 1, 2, 3, \ldots)
\]

By superposition, a formal solution to \( \text{\textcircled{1}} - \text{\textcircled{2}} - \text{\textcircled{3}} - \text{\textcircled{4}} - \text{\textcircled{5}} - \text{\textcircled{6}} \) is
\[
    u(x, y, z) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \Xi_\ell \Upsilon_m \Psi \sin(\ell \pi x) \sin(m \pi y) \sinh(\pi z \sqrt{\ell^2 + m^2})
\]

where the \( \Xi_\ell \Upsilon_m \Psi \) are arbitrary constants. To satisfy \( \text{\textcircled{7}} \) we must have...
\[ f(x,y) = u(x, y, 1) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} b_{lm} \sin(l\pi x) \sin(m\pi y) \]

for all \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \); here \( b_{lm} = c_{lm} \sinh(\pi \sqrt{l^2 + m^2}) \) for \( l = 1, 2, 3, \ldots \) and \( m = 1, 2, 3, \ldots \). But the \( b_{lm} \) must be the Fourier coefficients of the function \( g(x,y) = 4x(1-x)y(1-y) \) on the unit square \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \), with respect to the orthogonal set

\[ \phi_{lm}(x,y) = \sin(l\pi x)\sin(m\pi y) \quad (l = 1, 2, 3, \ldots; m = 1, 2, 3, \ldots) \]

on the unit square. That is,

\[
\begin{align*}
b_{lm} & = \frac{\langle g, \phi_{lm} \rangle}{\langle \phi_{lm}, \phi_{lm} \rangle} = b_{lm} = \frac{\int_0^1 \int_0^1 4x(1-x)y(1-y)\sin(l\pi x)\sin(m\pi y) \, dx \, dy}{\int_0^1 \int_0^1 \sin^2(l\pi x)\sin^2(m\pi y) \, dx \, dy} \\
& = \left( \frac{\int_0^1 4x(1-x)\sin(l\pi x) \, dx}{\int_0^1 \sin^2(l\pi x) \, dx} \right) \cdot \left( \frac{\int_0^1 4y(1-y)\sin(m\pi y) \, dy}{\int_0^1 \sin^2(m\pi y) \, dy} \right) \\
& = \left( 2\int_0^1 4x(1-x)\sin(l\pi x) \, dx \right) \cdot \left( 2\int_0^1 4y(1-y)\sin(m\pi y) \, dy \right)
\end{align*}
\]

By problem 5 we have

\[
\begin{align*}
2\int_0^1 4x(1-x)\sin(l\pi x) \, dx & = \begin{cases} 
0 & \text{if } l = 2j \text{ is even,} \\
\frac{32}{\pi^3(2j-1)^3} & \text{if } l = 2j-1 \text{ is odd,}
\end{cases} \\
2\int_0^1 4y(1-y)\sin(m\pi y) \, dy & = \begin{cases} 
0 & \text{if } m = 2k \text{ is even,} \\
\frac{32}{\pi^3(2k-1)^3} & \text{if } m = 2k-1 \text{ is odd.}
\end{cases}
\end{align*}
\]

Therefore
\[ b_{2j-1,2k-1} = \frac{(3z)}{\pi^2(2j-1)^3(2k-1)^3} \quad (j=1,3,5,... \text{ and } k=1,3,5,...) \text{ and all other } b_{k,m} \text{ are zero. Consequently,} \]

\[ c_{2j-1,2k-1} = \frac{(3z)^2}{\pi^2(2j-1)^3(2k-1)^3 \sinh(\sqrt{(2j-1)^2+(2k-1)^2})} \quad (j=1,3,5,... \text{ and } k=1,3,5,...) \]

and all other \( c_{k,m} \) are zero. Thus a solution to (1)-(5)-(3)-(4)-(5)-(6)-(7) is

\[ u(x,y,z) = \frac{1024}{\pi^6} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sin((2j-1)\pi x)\sin((2k-1)\pi y)\sinh(\pi \sqrt{(2j-1)^2+(2k-1)^2})}{(2j-1)^3(2k-1)^3 \sinh(\pi \sqrt{(2j-1)^2+(2k-1)^2})} \]
Number taking exam: 36

Mean: 131.0
Median: 140
Standard deviation: 37.1

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