1. (25 pts.) Solve \[ t \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial t} = 0 \] subject to \( u(x, 0) = x^4 \) for all real \( x \).

The solutions to \( t \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial t} = 0 \) are constant along the characteristic curves satisfying \( \frac{dt}{dx} = \frac{b(x,t)}{a(x,t)} = \frac{x}{t} \). Thus \( \frac{t^2}{2} = \int t \, dt = \int x \, dx = \frac{x^2}{2} + \zeta \).

So \( t^2 - x^2 = c \) are the characteristic curves (\( c = 2\zeta \) is constant). Therefore along a characteristic curve

\[ u(x,t) = u(x,t(x)) = u(0,t(0)) = u(0, t\sqrt{c}) = f(c) \]

The general solution to the P.D.E. is \( u(x,t) = f(t^2 - x^2) \) where \( f \) is a differentiable function of a single real variable. The need to choose \( f \) so the auxiliary condition is satisfied:

\[ x^4 = u(x,0) = f(-x^2) \quad \text{for all real } x. \]

I.e. \( (x^2)^2 = f(-x^2) \) for all real \( x \) implies \( f(z) = z^2 \) for all \( z \leq 0 \).

The solution is \( u(x,t) = (t^2 - x^2)^2 \) and it is unique in the region \( t^2 - x^2 < 0 \).
2. (25 pts.) Solve \( u_{xx} + u_{xy} - 20u_{yy} = 0 \) in the \( xy \)-plane, subject to \( u(x,0) = 25x^2 + 64x^3 \) and \( u_y(x,0) = -10x + 48x^2 \) for all real \( x \).

\[
0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 20 \frac{\partial^2 u}{\partial y^2} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} - 20 \frac{\partial^2}{\partial y^2} \right) u = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + 4 \frac{\partial}{\partial y} \right) u
\]

Let \[ \begin{cases} \xi = 5x - xy = 5x - y \\ \eta = (y - 4y) = 4x + y \end{cases} \]

Then \[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial}{\partial \xi} + 4 \frac{\partial}{\partial \eta} \right) u \]

Therefore, as operators,

\[
\frac{\partial}{\partial x} + 5 \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + 5 \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) = 9 \frac{\partial}{\partial \eta},
\]

\[
\frac{\partial}{\partial x} - 4 \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 4 \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) = 0 \frac{\partial}{\partial \xi},
\]

and so the P.D.E. is transformed into the equivalent equation

\[
0 = \left( 9 \frac{\partial}{\partial \eta} \right) \left( 9 \frac{\partial}{\partial \xi} \right) u = 81 \frac{\partial^2 u}{\partial \eta \partial \xi}.
\]

The general solution is thus \( u = f(\xi) + g(\eta) \) where \( f \) and \( g \) are arbitrary \( C^2 \)-functions of a single real variable. Then

\[
(*) \quad u(x,y) = f(5x-y) + g(4x+y)
\]

is the general solution of (1) in the \( xy \)-plane.

Applying (3) to \( \frac{\partial u}{\partial y} = -f'(5x-y) + g'(4x+y) \) yields \( -f'(5x) + g'(4x) = 10x + 48 \) (4)

Applying (2) to (4) gives \( f(5x) + g(4x) = u(x,0) = 25x^2 + 64x^3 \), (5)

and differentiating yields \( 5f'(5x) + 4g'(4x) \overset{\oplus}{=} 50x + 19 \)

Multiplying (4) by \( 5 \) and adding the result to (5) produces \( 9g'(4x) = 432x^2 \) or

\( \text{OVER} \)
\[ g'(4x) = 48x^2 = 3(4x)^2 \text{ for all real } x \]

or \[ g'(z) = 3z^2 \text{ for all real } z. \]

Thus \( g(z) = z^3 + c. \) Substituting in (†) yields

\[ f(5x) + (4x) + c = 25x^2 + 64x^3 \]

which implies \( f(5x) = 25x^2 - c = (5x)^2 - c. \) That is \( f(w) = w^2 - c \)

for all real \( w. \) Therefore

\[ u(x, y) = f(5x - y) + g(4x + y) = (5x - y)^2 + (4x + y)^3 \]

so

\[ u(x, y) = (5x - y)^2 + (4x + y)^3 \]

solves our problem.
3. (25 pts.) A homogeneous solid material occupying $D = \{(x, y, z) \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 100\}$ is completely insulated and its initial temperature at position $(x, y, z)$ in $D$ is $200/\sqrt{x^2 + y^2 + z^2}$.

(a) Write (without proof or derivation) the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x, y, z, t)$ at position $(x, y, z)$ in $D$ and time $t \geq 0$.

(b) Use the divergence theorem to help show that the heat energy $H(t) = \iiint_D cp u(x, y, z, t) dV$ of the material in $D$ at time $t$ is a constant function of time. Here $c$ and $\rho$ denote the (constant) specific heat and mass density, respectively, of the material in $D$.

(c) Compute the (constant) steady-state temperature that the material in $D$ reaches after a long time.

\[
\begin{align*}
\text{(a)} & \quad \frac{\partial u}{\partial t} - k \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) = 0 & \text{if } 4 < x^2 + y^2 + z^2 < 100 \text{ and } t > 0 , \\
& \quad \frac{\partial u}{\partial n} = 0 & \text{if } x^2 + y^2 + z^2 = 4 \text{ or } x^2 + y^2 + z^2 = 100 \text{ and } t > 0 , \\
& \quad u(x, y, z, 0) = \frac{200}{\sqrt{x^2 + y^2 + z^2}} & \text{if } 4 \leq x^2 + y^2 + z^2 \leq 100 . \\
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad \frac{dH}{dt} = \frac{1}{dt} \iiint_D cp \frac{\partial u}{\partial t} (x, y, z, t) dV = \iiint_D cp \frac{\partial u}{\partial t} (x, y, z, t) dV \\
& \quad = \iiint_D cp k \nabla \cdot \nabla u (x, y, z, t) dV = \iiint_D cp k \nabla \cdot \nabla u (x, y, z, t) dV = \iiint_D cp k \partial u / \partial n dS = 0 .
\end{align*}
\]

Therefore $H = H(t)$ is constant for $t \geq 0$.

\[
\begin{align*}
\text{(c)} & \quad \text{Assume } U \text{ is the (constant) steady-state temperature that the material in } D \text{ reaches; i.e., } U = \lim_{t \to \infty} u(x, y, z, t) \text{ at all points } (x, y, z) \text{ in } D \text{. By part (b), } H(0) = H(t) \text{ for all } t \geq 0 \text{ so}
\end{align*}
\]

\[
\begin{align*}
H(0) = \lim_{t \to \infty} H(t) &= \lim_{t \to \infty} \iiint_D cp u(x, y, z, t) dV = \iiint_D cp \lim_{t \to \infty} u(x, y, z, t) dV \\
&= \iiint_D cp U dV = cp U \text{ vol}(D). \quad \text{But (OVER)}
\end{align*}
\]
\[ \text{vol}(D) = \iiint_D dV = \int_0^\pi \int_0^2 \int_0^{10} r^2 \sin \phi \, dr \, d\phi \, d\theta = \frac{4\pi}{3} r^3 \bigg|_0^{10} = \frac{4\pi}{3} (1000-8) \]

and

\[ H(0) = \iiint_D c_p u(x, y, z, 0) \, dV = \iiint_D c_p \frac{200}{\sqrt{x^2 + y^2}} \, dV = \int_0^\pi \int_0^2 c_p \frac{200}{r} \cdot r^2 \sin \phi \, dr \, d\phi \bigg|_0^{10} = \int_0^{2\pi} \int_0^\pi c_p 100(100-4) (-\cos \phi) \, d\phi \, d\theta \bigg|_0^{\pi} = 4\pi c_p 100(100-4). \]

Therefore

\[ U = \frac{H(0)}{2 \cdot \text{vol}(D)} = \frac{4\pi c_p 100(100-4)}{c_p \frac{4\pi}{3} (1000-8)} = \frac{300(100-4)}{1000-8} = \frac{28800}{.992} = \frac{900}{31} \]

\( \approx 29. \)
4. (25 pts.) Solve \( u_x - u_{yy} = 0 \) for \( -\infty < y < \infty \) and \( 0 < x < \infty \), subject to \( u(0, y) = e^{3y} \) if \( -\infty < y < \infty \).

Recall that a solution to
\[
  u_t - ku_{xx} = 0 \quad \text{if} \quad -\infty < x < \infty, \quad 0 < t < \infty,
\]
\[
  u(x, 0) = \varphi(x) \quad \text{if} \quad -\infty < x < \infty,
\]
is given by
\[
  u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \varphi(s) \, ds \quad (t > 0) \text{ provided}
\]
\( \varphi \) is continuous and bounded on \(( -\infty, \infty ) \). Our problem has the same form under the correspondence \(( x, t ) \leftrightarrow ( y, x ) \), except that \( \varphi(y) = e^{3y} \) is not bounded on \(-\infty < y < \infty \). Therefore a candidate for a solution to our problem is

\[
  u(x, y) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{4x}} \varphi(s) \, ds \quad (x > 0, \quad k = 1, \quad \varphi(s) = e^{3s})
\]

\[
  = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{4x}} \cdot e^{3s} \, ds
\]

\[
  = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2-12xs}{4x}} \, ds.
\]

Completing the square in \( s \) in the exponent yields

\[
  (y-s)^2 - 12xs = s^2 - 2ys + y^2 - 12xs
\]

\[
  = s^2 - 2s(y+6x) + (y+6x)^2 - (y+6x)^2 + y^2
\]

\[
  = (s - (y+6x))^2 - (y^2 + 12xy + 36x^2) + y^2
\]

\[
  = (s - (y+6x))^2 - 12x(y+3x).
\]

Therefore

(over)
\[ u(x,y) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(s-(y+6x))^2}{4x}} ds \]

\[ = \frac{e^{3(y+3x)}}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(s-(y+6x))^2}{4x}} ds \]

Let \( p = \frac{s-(y+6x)}{\sqrt{4x}} \). Then \( dp = \frac{ds}{\sqrt{4x}} \) and \( \left\{ \begin{array}{l} p \to \infty \iff s \to \infty \\ p \to -\infty \iff s \to -\infty \end{array} \right. \)

Then,

\[ u(x,y) = \frac{e^{3(y+3x)}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp \]

\[ u(x,y) = e^{3(y+3x)} \]

Since \( p \) was unbounded we need to check this result.

\[ u(0,y) = e^{3y} \quad \text{if} \quad -\infty < y < \infty \]

\[ \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 9e^{3(y+3x)} - 9e^{3(y+3x)} = 0 \quad (\text{in the entire } xy \text{-plane, in fact}) \]

Therefore \( u(x,y) = e^{3y+9x} \) solves the IVP.
5.(25 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem: $u_t - ku_{xx} = f(x,t)$ for $-\infty < x < \infty$, $0 < t < \infty$, and $u(x,0) = \phi(x)$ if $-\infty < x < \infty$.

Suppose $u = u(x,t)$ solves $\odot$. Take the Fourier transform of $\odot$. To obtain

\[ \frac{\partial}{\partial t} \mathcal{F}(u)(\xi) + k^2 \mathcal{F}(u)(\xi) = \mathcal{F}(u_t - ku_{xx})(\xi) = f(\xi, t), \]

where $\mathcal{F}$ is the Fourier transform. Integrating factor of this first-order linear ODE in $t$ (with parameter $\xi$) is

\[ e^{\int k^2 dt} = e^{k^2 t}. \]

Multiplying the above ODE by the integrating factor gives

\[ \frac{\partial}{\partial t} \left( e^{k^2 t} \mathcal{F}(u)(\xi) \right) = e^{k^2 t} \frac{\partial}{\partial t} \mathcal{F}(u)(\xi) + k^2 e^{k^2 t} \mathcal{F}(u)(\xi) = e^{k^2 t} f(\xi, t), \]

so

\[ e^{k^2 t} \mathcal{F}(u)(\xi) = c(\xi) + \int_0^t e^{k^2 \tau} f(\xi, \tau) d\tau \]

and hence

\[ \mathcal{F}(u)(\xi) = c(\xi) e^{-k^2 t} + \int_0^t e^{-k^2 (t-\tau)} f(\xi, \tau) d\tau. \]

Applying \(\odot\) in the above identity, gives $c(\xi) = \hat{\phi}(\xi)$. Thus

\[ \mathcal{F}(u)(\xi) = e^{-k^2 t} \hat{\phi}(\xi) + \int_0^t e^{-k^2 (t-\tau)} \mathcal{F}(f(\xi, \tau)) d\tau. \]

Taking $a = \frac{1}{4kt}$ in the table entry $\#4: \mathcal{F}(\frac{a(x)}{\sqrt{2\pi}})(\xi) = e^{-\frac{\xi^2}{4a}}$

we have

\[ \mathcal{F} \left( \frac{1}{\sqrt{2\pi kt}} e^{-\frac{\xi^2}{4kt}} \right)(\xi) = e^{-k^2 t}, \]

and similarly

\[ \mathcal{F} \left( \frac{1}{\sqrt{2\pi k(t-\tau)}} e^{-\frac{(\xi-\frac{\xi^2}{4k(t-\tau)}}{2}} \right)(\xi) = e^{-k^2 (t-\tau)}, \]

Thus

\[ (\text{OVER}) \]
\[
\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{-\xi^2}{4\pi t}\right) \phi(\xi) + \int_0^t \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]

But \( \hat{g}(\xi) = \mathcal{F}\left(\frac{\delta(\xi)}{\sqrt{4\pi t}}\right) \phi(\xi) \).

\[
\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{-\xi^2}{4\pi t}\right) \phi(\xi) + \int_0^t \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]

Integrating the order of integration in the second term of the right member of the equation above gives

\[
\int_0^t \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau = \int_0^{\infty} \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{\xi^2}{4\pi (t-\tau)}} f(\xi) d\tau \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\pi (t-\tau)}} dx
\]

Substituting this in above gives

\[
\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{-\xi^2}{4\pi t}\right) \phi(\xi) + \int_0^{\infty} \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]

The inversion theorem then gives

\[
\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{-\xi^2}{4\pi t}\right) \phi(\xi) + \int_0^{\infty} \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]

\[
u(x,t) = \int_{-\infty}^{\infty} \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]

\[
u(x,t) = \int_{-\infty}^{\infty} \mathcal{F}\left(\frac{-\xi^2}{4\pi (t-\tau)}\right) f(\xi) d\tau
\]
6 (25 pts.) (a) Find a solution to \( u_{xx} + u_{yy} = 0 \) for \( 0 < x < 1 \) and \( 0 < y < 1 \), subject to the boundary conditions \( u(x,0) = u(x,1) = 0 \) and \( u(0,y) = u(1,y) = 3\sin(\pi y) - \sin(3\pi y) \) if \( 0 \leq y \leq 1 \).

(b) Show that the solution to the problem in part (a) is unique.

\[
\begin{align*}
7. & \quad \Delta \text{ we use separation of variables. The seek nontrivial solutions of } 0 - \text{(1) - (2) - (3) - (4)} \text{ of the form } \Sigma(x) Y(y) = u(x,y). \text{ Substituting in (1) gives } \Sigma''(x) Y(y) + \Sigma(x) Y''(y) = 0 \text{ for all } (x,y) \text{ in the unit square. Rearranging } + \frac{\Sigma''(x)}{\Sigma(x)} = -\frac{Y''(y)}{Y(y)} = \text{constant} = \lambda. \text{ Substituting in (2), (3), and (4) and using the fact that } u(x,y) \text{ is not identically zero we see that } \\
& \Sigma(x) Y(0) = 0 \text{ and } \Sigma(x) Y(1) \text{ for all } 0 \leq x \leq 1 \text{ implies } Y(0) = Y(1) = 0 \text{ and } \\
& \Sigma(0) Y(y) = 0 \text{ for all } 0 \leq y \leq 1 \text{ implies } \Sigma(0) = 0. \text{ Therefore we have the coupled system } \\
& \begin{cases}
\Sigma''(x) - \lambda \Sigma(x) = 0 \text{ if } 0 < x < 1 \text{ and } \Sigma(0) = 0 \\
Y''(y) + \lambda Y(y) = 0 \text{ if } 0 < y < 1 \text{ and } Y(0) = Y(1) = 0
\end{cases}
\end{align*}
\]

1. Since the operator \( T = \frac{d^2}{dy^2} \) is symmetric on \( V_D = \{ f \in C^2([0,1]) : f(0) = f(1) \} \), it follows that the eigenvalues are real. Since \( f(0)f'(0) - f(1)f'(1) = 0 \) for all \( f \in V_D \), it follows that the eigenvalues are all positive.

Choose \( \lambda = \beta^2 \), where \( \beta > 0 \). The must solve \( \Sigma''(y) + \beta^2 \Sigma(y) \) is subject to \( \Sigma(0) = 0 \).

The general solution of the DE is \( \Sigma(y) = c_1 \cos(\beta y) + c_2 \sin(\beta y) \). Applying the B.C.'s we see that \( 0 = c_1 \) and \( 0 = c_1 \cos(\beta) + c_2 \sin(\beta) \). Therefore \( c_1 = 0 \) and \( \beta = \beta_n = n\pi \text{ (n=1,2,\ldots)} \) for nontrivial solutions. Hence, \( \lambda_n = (n\pi)^2 \) and \( Y_n(y) = \sin(n\pi y) \) are the eigenvalues and eigenfunctions, respectively (n=1,2,3,\ldots).

The solution of the x-equations: \( \Sigma_n''(x) - (n\pi)^2 \Sigma_n(x) = 0, \Sigma_n(0) = 0 \), are (up to a constant factor) \( \Sigma_n(x) = \sin(n\pi x) \).

By the superposition principle \( u(x,y) = \sum_{n=1}^{\infty} c_n \Sigma_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \sin(n\pi y) \)

is a formal solution to 0 - (1) - (2) - (3) - (4) for any choice of constants \( c_1, c_2, c_3, \ldots \)

(over)
We want to choose the constants so (5) is satisfied:

$$3 \sin(\pi y) - \sin(3\pi y) = u(1, y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi) \sin(n\pi y) \quad \text{for all } 0 \leq y \leq 1$$

That is:

$$3 \sin(\pi y) - \sin(3\pi y) = c_1 \sinh(\pi) \sin(\pi y) + c_2 \sinh(2\pi) \sin(2\pi y) + c_3 \sinh(3\pi) \sin(3\pi y) + \ldots$$

for all $0 \leq y \leq 1$. By inspection, a solution occurs if we choose

$$c_1 = \frac{3}{\sinh(\pi)}, \quad c_2 = \frac{-1}{\sinh(2\pi)}, \quad \text{and } c_n = 0 \text{ for all other } n.$$  

Therefore

$$u(x, y) = \frac{3 \sinh(\pi x) \sin(\pi y)}{\sinh(\pi)} - \frac{\sinh(3\pi x) \sin(3\pi y)}{\sinh(3\pi)}$$

solves (1)-(5). (Continuous)

(b) Suppose that $v = v(x, y)$ is another solution to (1)-(2)-(3)-(4)-(5). Then the continuous function $w(x, y) = u(x, y) - v(x, y)$ solves $w_{xx} + w_{yy} = 0$ in the unit square $S = \{(x, y): 0 < x < 1, 0 < y < 1\}$ and $w(x, y) = 0$ if $(x, y) \in \partial S$, the boundary of $S$. By the maximum/minimum principle for continuous solutions to Laplace’s equation

$$0 = \min_{(x, y) \in \partial S} w(x, y) = \min_{(x, y) \in S} w(x, y) \leq \max_{(x, y) \in S} w(x, y) = \max_{(x, y) \in \partial S} w(x, y) = 0,$$

That is, $w(x, y) \equiv 0$ on $S$, so $u(x, y) = v(x, y)$ for all $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This shows that there is only one solution to (1)-(2)-(3)-(4)-(5).
7. (25 pts.) Consider the constant function \( \phi(x) = 100 \) on the closed interval \([0, 1]\).

(a) Show that the Fourier sine series of \( \phi \) on \([0, 1]\) is
\[
\sum_{k=0}^{\infty} \frac{400 \sin((2k+1)\pi x)}{2k+1}.
\]

(b) Does the Fourier sine series of \( \phi \) converge to \( \phi \) in the \( L^2 \) sense on \([0, 1]\)? Justify your answer.

(c) For which \( x \) in \([0, 1]\) does the Fourier sine series of \( \phi \) converge pointwise to \( \phi(x) \)? Justify your answer.

(d) Does the Fourier sine series of \( \phi \) converge to \( \phi \) uniformly on \([0, 1]\)? Justify your answer.

(e) Use the results above to help find the sum of the series
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.
\]

(f) Use the results above to help find the sum of the series
\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}. \quad \text{(Hint: Parseval's identity.)}
\]
(e) By part (c), \(100 = \frac{100}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi/2)}{2k+1}\)

Taking \(x = \frac{1}{2}\) in this identity yields

\[\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\]

(f) By Parseval's identity,

\[\sum_{n=1}^{\infty} |b_n|^2 \int_0^1 \sin^2(n\pi x) \, dx = \int_0^1 |f(x)|^2 \, dx\]

\[\sum_{k=0}^{\infty} \left(\frac{400}{(2k+1)\pi}\right)^2 \cdot \frac{1}{2} = 10,000\]

\[\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{2(10,000)\pi^2}{400^2}\]

\[\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}\]
8. (25 pts.) Consider a metal rod of length 1 meter, insulated along its sides but not at its ends, which is initially at temperature 100° C. Suddenly both ends are inserted in an ice bath of temperature 0° C.

(a) Write the partial differential equation, boundary conditions, and initial condition that model the temperature $u = u(x, t)$ of the rod at position $x$ in $[0,1]$ and time $t > 0$.

(b) Find a formula for the temperature $u = u(x, t)$. (Hint: You may find the results of problem 7 useful.)

\[
\begin{align*}
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{if } 0 < x < 1 \text{ and } 0 < t < \infty, \\
\frac{\partial u}{\partial x} &= 0 \quad \text{if } t > 0, \\
u(x, 0) &= 100 \quad \text{if } 0 \leq x \leq 1.
\end{align*}
\]

(b) We use separation of variables. We seek nontrivial solutions of (1)-(2)-(3) of the form $u(x,t) = \Xi(x)T(t)$. Then (1) implies $-\frac{T'(t)}{kT(t)} = -\frac{\Xi''(x)}{\Xi(x)} = \lambda$.

Also (2) and (3) imply $\Xi(0) = 0$ and $\Xi(1) = 0$ since $u(x,t) = \Xi(x)T(t)$ is not identically zero. Therefore

\[
\begin{align*}
\Xi''(x) + \lambda \Xi(x) &= 0, \\
\Xi(0) &= 0 = \Xi(1) \\
T'(t) + \lambda kT(t) &= 0.
\end{align*}
\]

By the work in problem 6, the eigenvalues and eigenfunctions are, respectively,$\lambda_n = (n\pi)^2$ and $\Xi_n(x) = \sin(n\pi x)$ ($n = 1, 2, 3, \ldots$). The solution to the $t$-equation $T'_n(t) + k(n\pi)^2 T_n(t) = 0$ is $T_n(t) = e^{-k(n\pi)^2 t}$ ($n = 1, 2, 3, \ldots$), at least up to a constant factor. By superposition,

\[
u(x, t) = \sum_{n=1}^{\infty} c_n \Xi_n(x)T_n(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)e^{-k(n\pi)^2 t}
\]

solves (1)-(2)-(3) formally for any choice of constants $c_1, c_2, c_3, \ldots$. We need to choose the constants so condition (4) is satisfied:

\[
100 = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \quad \text{for all } x \text{ in } (0, 1).
\]

(OVER)
By problem 7, we should choose

\[ c_n = \begin{cases} 
0 & \text{if } n = 2j \text{ is even}, \\
\frac{400}{(2j+1)\pi} & \text{if } n = 2j+1 \text{ is odd}.
\end{cases} \]

Then,

\[ u(x, t) = \sum_{j=0}^{\infty} \frac{400 \sin((2j+1)\pi x)}{\pi (2j+1)} e^{-k(2j+1)^2 \pi^2 t} \]

solves (1)-(4).
A Brief Table of Fourier Transforms

\[ f(x) \quad \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx \]

A. \[
\begin{cases}
1 & \text{if } -b < x < b, \\
0 & \text{otherwise.}
\end{cases}
\]

\[ \frac{2 \sin(b\xi)}{\pi} \frac{\sqrt{2}}{\xi} \]

B. \[
\begin{cases}
1 & \text{if } c < x < d, \\
0 & \text{otherwise.}
\end{cases}
\]

\[ e^{-ic\xi} - e^{-id\xi} \]

\[ \frac{i\xi\sqrt{2\pi}}{\xi} \]

C. \[
\frac{1}{x^2 + a^2} \quad (a > 0)
\]

\[ \frac{\pi}{2} \frac{e^{-a|\xi|}}{a} \]

D. \[
\begin{cases}
x & \text{if } 0 < x < b, \\
2b - x & \text{if } b < x < 2b, \\
0 & \text{otherwise.}
\end{cases}
\]

\[ -1 + 2e^{-ib\xi} - e^{-2ib\xi} \]

\[ \frac{\xi^2}{2\pi} \]

E. \[
\begin{cases}
e^{-ax} & \text{if } x > 0, \\
0 & \text{otherwise.}
\end{cases} \quad (a > 0)
\]

\[ \frac{1}{(a + i\xi)\sqrt{2\pi}} \]

F. \[
\begin{cases}
e^{ax} & \text{if } b < x < c, \\
0 & \text{otherwise.}
\end{cases}
\]

\[ e^{(a-i\xi)c} - e^{(a-i\xi)b} \]

\[ (a - i\xi)\sqrt{2\pi} \]

G. \[
\begin{cases}
e^{iax} & \text{if } -b < x < b, \\
0 & \text{otherwise.}
\end{cases}
\]

\[ \frac{2\sin(b(\xi-a))}{\pi} \frac{\sqrt{2}}{\xi - a} \]

H. \[
\begin{cases}
e^{iax} & \text{if } c < x < d, \\
0 & \text{otherwise.}
\end{cases}
\]

\[ \frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi} \]

I. \[
\begin{cases}
e^{-ax^2} & \text{(a > 0)}
\end{cases}
\]

\[ \frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)} \]

J. \[
\begin{cases}
\frac{\sin(ax)}{x} & \text{(a > 0)}
\end{cases}
\]

\[ \begin{cases} 0 & \text{if } |\xi| \geq a, \\
\frac{1}{\sqrt{\pi/2}} & \text{if } |\xi| < a. \end{cases} \]
\[ X'' + \lambda X = 0 \] in \((a, b)\) with any \textit{symmetric} BC. \hspace{1cm} (1)

Now let \(f(x)\) be any function defined on \(a \leq x \leq b\). Consider the Fourier series for the problem (1) with any given boundary conditions that are \textit{symmetric}. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

**Theorem 2. Uniform Convergence**  
The Fourier series \(\sum A_n X_n(x)\) converges to \(f(x)\) uniformly on \([a, b]\) provided that

(i) \(f(x), f'(x), \) and \(f''(x)\) exist and are continuous for \(a \leq x \leq b\) and

(ii) \(f(x)\) satisfies the given boundary conditions.

**Theorem 3. L^2 Convergence**  
The Fourier series converges to \(f(x)\) in the mean-square sense in \((a, b)\) provided only that \(f(x)\) is any function for which

\[ \int_a^b |f(x)|^2 \, dx \] is finite. \hspace{1cm} (8)

**Theorem 4. Pointwise Convergence of Classical Fourier Series**

(i) The classical Fourier series (full or sine or cosine) converges to \(f(x)\) pointwise on \((a, b)\), provided that \(f(x)\) is a continuous function on \(a \leq x \leq b\) and \(f'(x)\) is piecewise continuous on \(a \leq x \leq b\).

(ii) More generally, if \(f(x)\) itself is only piecewise continuous on \(a \leq x \leq b\) and \(f'(x)\) is also piecewise continuous on \(a \leq x \leq b\), then the classical Fourier series converges at every point \(x (-\infty < x < \infty)\). The sum is

\[ \sum_n A_n X_n(x) = \frac{1}{2} [f(x+) + f(x-)] \] for all \(-\infty < x < \infty\). The sum is \[ \frac{1}{2} [f_{\text{ext}}(x+) + f_{\text{ext}}(x)] \] for all \(-\infty < x < \infty\), where \(f_{\text{ext}}(x)\) is the extended function (periodic, odd periodic, or even periodic).

**Theorem 4.**  
If \(f(x)\) is a function of period \(2l\) on the line for which \(f(x)\) and \(f'(x)\) are piecewise continuous, then the classical full Fourier series converges to \[ \frac{1}{2} [f(x+) + f(x-)] \] for \(-\infty < x < \infty\).
Math 325  
Summer 2009  
Final Exam

\[ n = 16 \]
\[ \mu = 141.0 \]
\[ \sigma = 30.2 \]

<table>
<thead>
<tr>
<th>Distribution of Scores:</th>
<th>Frequency</th>
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<tr>
<td>146 - 173</td>
<td>B</td>
</tr>
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<td>C (graduate), B (undergraduate)</td>
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<tr>
<td>100 - 119</td>
<td>C</td>
</tr>
<tr>
<td>0 - 99</td>
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\[ \text{frequency} \]