1.(25 pts.) Solve $t\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial t}=0$ subject to $u(x,0)=x^4$ for all real x

The solutions to $t\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial t}=0$ are constant along the characteristic curves

6 satisfying $\dfrac{dt}{dx}=\dfrac{b(x,t)}{a(x,t)}=\dfrac{x}{t}$. Thus $\dfrac{t^2}{2}=\int t\,dt=\int x\,dx=\dfrac{x^2}{2}+\tilde{c}$

12 so $t^2-x^2=c$ are the characteristic curves $(c=2\tilde{c}=\text{constant})$. Therefore along a characteristic curve

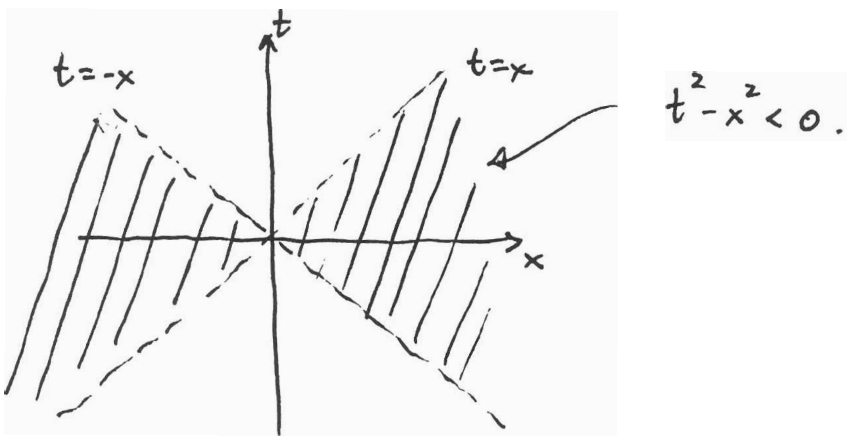
$$u(x,t)=u(x,t(x))=u(0,t(0))=u(0,\pm\sqrt{c})=f(c).$$

18 The general solution to the P.D.E. is $u(x,t)=f(t^2-x^2)$ where f is a differentiable function of a single real variable. We need to choose f so the auxiliary condition is satisfied:

$$x^4=u(x,0)=f(-x^2) \quad \text{for all real } x.$$

22 I.e. $(-x^2)^2=f(-x^2)$ for all real x implies $f(z)=z^2$ for all $z\le 0$.

25 The solution is $\boxed{u(x,t)=(t^2-x^2)^2}$ and it is unique in the region $t^2-x^2<$



2.(25 pts.) Solve $u_{xx}+u_{xy}-20u_{yy}\overset{①}{=}0$ in the xy-plane, subject to $u(x,0)\overset{②}{=}25x^2+64x^3$ and $u_y(x,0)\overset{③}{=}-10x+48x^2$ for all real x.

$$0=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}-20\frac{\partial^2 u}{\partial y^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x\partial y}-20\frac{\partial^2}{\partial y^2}\right)u=\overbrace{\left(\frac{\partial}{\partial x}+5\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-4\frac{\partial}{\partial y}\right)}^{\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)\left(\gamma\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial y}\right)}u$$

Let $\begin{cases}\xi=\beta x-\alpha y=5x-y\\ \eta=-(\gamma x-\delta y)=4x+y\end{cases}$. Then $\dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial x}+\dfrac{\partial v}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial x}=\left(5\dfrac{\partial}{\partial \xi}+4\dfrac{\partial}{\partial \eta}\right)v$

and $\dfrac{\partial v}{\partial y}=\dfrac{\partial v}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial y}+\dfrac{\partial v}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial y}=\left(-\dfrac{\partial}{\partial \xi}+\dfrac{\partial}{\partial \eta}\right)v$

Therefore, as operators,

$$\frac{\partial}{\partial x}+5\frac{\partial}{\partial y}=5\frac{\partial}{\partial \xi}+4\frac{\partial}{\partial \eta}+5\left(-\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)=9\frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial x}-4\frac{\partial}{\partial y}=5\frac{\partial}{\partial \xi}+4\frac{\partial}{\partial \eta}-4\left(-\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)=9\frac{\partial}{\partial \xi},$$

and so the P.D.E. ① is transformed into the equivalent equation

$$0=\left(9\frac{\partial}{\partial \eta}\right)\left(9\frac{\partial}{\partial \xi}\right)u=81\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right).$$

The general solution is thus $u=f(\xi)+g(\eta)$ where f and g are arbitrary C^2-functions of a single real variable. Then

$$(*)\qquad u(x,y)=f(5x-y)+g(4x+y)$$

is the general solution of ① in the xy-plane.

Applying ③ to $\dfrac{\partial u}{\partial y}\overset{(*)}{=}-f'(5x-y)+g'(4x+y)$ yields $-f'(5x)+g'(4x)\overset{④}{=}-10x+48$

Applying ② to (*) gives $f(5x)+g(4x)=u(x,0)=25x^2+64x^3$, (†)

and differentiating yields $\dashrightarrow$ $5f'(5x)+4g'(4x)\overset{⑤}{=}50x+19$

Multiplying ④ by 5 and adding the result to ⑤ produces $9g'(4x)=432x^2$ or

(OVER)

$$g'(4x) = 48x^2 = 3(4x)^2 \quad \text{for all real } x$$

or $g'(z) = 3z^2$ for all real z.

19 Thus $g(z) = z^3 + c$. Substituting in $(\dagger)$ yields

$$f(5x) + (4x)^3 + c = 25x^2 + 64x^3$$

22 which implies $f(5x) = 25x^2 - c = (5x)^2 - c$. That is $f(w) = w^2 - c$ for all real w. Therefore

$$u(x,y) = f(5x-y) + g(4x+y) = (5x-y)^2 - \cancel{c} + (4x+y)^3 + \cancel{c}$$

25 so $\boxed{u(x,y) = (5x-y)^2 + (4x+y)^3}$ solves our problem.

3.(25 pts.) A homogeneous solid material occupying $D=\{(x,y,z)\in\mathbb{R}^3 : 4\le x^2+y^2+z^2\le 100\}$ is completely insulated and its initial temperature at position (x,y,z) in D is $200/\sqrt{x^2+y^2+z^2}$.

(a) Write (without proof or derivation) the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t\ge 0$.

(b) Use the divergence theorem to help show that the heat energy $H(t)=\iiint_D c\rho u(x,y,z,t)\,dV$ of the material in D at time t is a constant function of time. Here c and ρ denote the (constant) specific heat and mass density, respectively, of the material in D.

(c) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

9

(a)
$$\begin{cases} \dfrac{\partial u}{\partial t} - k\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \overset{①}{=} 0 & \text{if } 4<x^2+y^2+z^2<100 \text{ and } t>0, \\ \dfrac{\partial u}{\partial n} \overset{②}{=} 0 & \text{if } x^2+y^2+z^2=4 \text{ or } x^2+y^2+z^2=100 \text{ and } t\ge 0, \\ u(x,y,z,0) \overset{③}{=} \dfrac{200}{\sqrt{x^2+y^2+z^2}} & \text{if } 4\le x^2+y^2+z^2\le 100. \end{cases}$$

8

(b)
$$\frac{dH}{dt} = \frac{d}{dt}\iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D c\rho \frac{\partial u}{\partial t}(x,y,z,t)\,dV$$
$$\overset{①}{=} \iiint_D c\rho k\,\nabla\cdot(\nabla u)\,dV \overset{\text{divergence theorem}}{=} \iint_{\partial D} c\rho k(\nabla u\cdot\vec{n})\,dS = \iint_{\partial D} c\rho k\frac{\partial u}{\partial n}\,dS \overset{②}{=} 0.$$

Therefore $H=H(t)$ is constant for $t\ge 0$.

8

(c) Assume U is the (constant) steady-state temperature that the material in D reaches; i.e. $U=\lim_{t\to\infty} u(x,y,z,t)$ at all points (x,y,z) in D. By part (b), $H(0)=H(t)$ for all $t\ge 0$ so

$$H(0)=\lim_{t\to\infty}H(t) = \lim_{t\to\infty}\iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D c\rho \lim_{t\to\infty} u(x,y,z,t)\,dV$$

3 pts.

$$= \iiint_D c\rho U\,dV = c\rho U\,\mathrm{vol}(D).$$ But

(OVER)

2 pts. $$\mathrm{vol}(D) = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{10} r^2 \sin\varphi \, dr\, d\varphi\, d\theta = \frac{4\pi}{3} r^3 \Big|_2^{10} = \frac{4\pi}{3}(1000-8)$$

and

2 pts. $$H(0) = \iiint_D c\rho\, u(x,y,z,0)\, dV \overset{③}{=} \iiint_D c\rho \frac{200}{\sqrt{x^2+y^2+z^2}}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{10} c\rho \frac{200}{r}\cdot r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} c\rho\, 100 r^2 \Big|_2^{10} \sin\varphi \, d\varphi\, d\theta = 2\pi c\rho\, 100(100-4)(-\cos\varphi)\Big|_0^{\pi} = 4\pi c\rho\, 100(100-4).$$

Therefore

1 pt. $$v = \frac{H(0)}{2\rho\, \mathrm{vol}(D)} = \frac{4\pi c\rho\, 100(100-4)}{c\rho \frac{4\pi}{3}(1000-8)} = \frac{300(100-4)}{1000-8} = \frac{28800}{992} = \boxed{\frac{900}{31}}$$

$$\cong 29.$$

4.(25 pts.) Solve $u_x - u_{yy} = 0$ for $-\infty < y < \infty$ and $0 < x < \infty$, subject to $u(0,y) = e^{3y}$ if $-\infty < y < \infty$.

Recall that a solution to

$$u_t - k u_{xx} = 0 \quad \text{if} \quad -\infty < x < \infty,\ 0 < t < \infty,$$

$$u(x,0) = \varphi(x) \quad \text{if} \quad -\infty < x < \infty,$$

is given by $u(x,t) = \dfrac{1}{\sqrt{4k\pi t}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \varphi(s)\,ds$ $(t > 0)$ *provided* φ *is continuous and bounded on* $(-\infty, \infty)$. *Our problem has the same form under the correspondence* $(x,t) \longleftrightarrow (y,x)$ *except that* $\varphi(y) = e^{3y}$ *is not bounded on* $-\infty < y < \infty$. *Therefore a candidate for a solution to our problem is*

4 $$u(x,y) = \frac{1}{\sqrt{4k\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{4kx}} \varphi(s)\,ds \qquad (x > 0,\ k = 1,\ \varphi(s) = e^{3s})$$

6 $$= \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2}{4x}} \cdot e^{3s}\,ds$$

$$= \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-s)^2 - 12xs}{4x}}\,ds.$$

Completing the square in s in the numerator of the exponent yields

$$\begin{aligned}
(y-s)^2 - 12xs &= s^2 - 2ys + y^2 - 12xs \\
&= s^2 - 2s(y+6x) + (y+6x)^2 - (y+6x)^2 + y^2 \\
&= (s-(y+6x))^2 - (\cancel{y^2} + 12xy + 36x^2) + \cancel{y^2} \\
&= (s-(y+6x))^2 - 12x(y+3x).
\end{aligned}$$

Therefore

(OVER)

$$u(x,y) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(s-(y+6x))^2 - 12x(y+3x)}{4x}}\, ds$$

13

$$= \frac{e^{3(y+3x)}}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(s-(y+6x))^2}{4x}}\, ds .$$

17 Let $p = \frac{s-(y+6x)}{\sqrt{4x}}$. Then $dp = \frac{ds}{\sqrt{4x}}$ and $\begin{cases} p \to \infty \iff s \to \infty \\ p \to -\infty \iff s \to -\infty \end{cases}$

Then

21

$$u(x,y) = \frac{e^{3(y+3x)}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\, dp$$

23

$$u(x,y) = e^{3(y+3x)} .$$

Since φ was unbounded we need to check this result.

✓ $u(0,y) = e^{3y}$ if $-\infty < y < \infty$

25

✓ $\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 9e^{3(y+3x)} - 9e^{3(y+3x)}$

$= 0$ (in the entire xy-plane, in fact).

Therefore $\boxed{u(x,y) = e^{3y+9x}}$ solves the IVP.

5.(25 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem: $u_t - ku_{xx} \overset{①}{=} f(x,t)$ for $-\infty < x < \infty$, $0 < t < \infty$, and $u(x,0) \overset{②}{=} \phi(x)$ if $-\infty < x < \infty$.

Suppose $u = u(x,t)$ solves ①. Take the Fourier transform of ① w.r.t. x to obtain

3 pts. to here. $$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2\mathcal{F}(u)(\xi) = \mathcal{F}(u_t - ku_{xx})(\xi) = \hat{f}(\xi,t).$$

An integrating factor of this first-order linear ODE in t (with parameter ξ) is

5 $\mu = e^{\int k\xi^2 dt} = e^{k\xi^2 t}$. Multiplying the above ODE by the integrating factor gives

$$\frac{\partial}{\partial t}\left(e^{k\xi^2 t}\mathcal{F}(u)(\xi)\right) = e^{k\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}(u)(\xi) = e^{k\xi^2 t}\hat{f}(\xi,t),$$

so $$e^{k\xi^2 t}\mathcal{F}(u)(\xi) = c(\xi) + \int_0^t e^{k\xi^2 \tau}\hat{f}(\xi,\tau)\,d\tau$$ and hence

10 $$\mathcal{F}(u)(\xi) = c(\xi)e^{-k\xi^2 t} + \int_0^t e^{-k\xi^2(t-\tau)}\hat{f}(\xi,\tau)\,d\tau.$$

Applying ②, after setting $t=0$ in the above identity, gives $c(\xi) = \hat{\varphi}(\xi)$. Thus

13 $$\mathcal{F}(u)(\xi) = e^{-k\xi^2 t}\hat{\varphi}(\xi) + \int_0^t e^{-k\xi^2(t-\tau)}\hat{f}(\xi,\tau)\,d\tau.$$

Taking $a = \frac{1}{4kt}$ in the Fourier transform table entry I: $\mathcal{F}\left(\sqrt{2a}\, e^{-a(\cdot)^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}}$,

we have

$$\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi) = e^{-k\xi^2 t},$$

and similarly

$$\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi) = e^{-k\xi^2(t-\tau)}.$$

Thus

(OVER)

$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{1}{\sqrt{2kt}} e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)\hat{\varphi}(\xi) + \int_0^t \mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\hat{f}(\xi,\tau)d$$

But $\hat{g}(\xi)\hat{h}(\xi) = \mathcal{F}\left(\frac{g*h}{\sqrt{2\pi}}\right)(\xi)$ so

18 $$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} * \varphi\right)(\xi) + \int_0^t \mathcal{F}\left(\frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau$$

Interchanging the order of integration in the second term of the right member of the equation above gives

$$\int_0^t \mathcal{F}\left(\frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(\xi)\,d\tau = \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{4k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(x)\, e^{-i\xi x}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \int_0^t \left(\frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(x)\,d\tau\,dx$$

$$= \mathcal{F}\left(\int_0^t \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\,d\tau\right)(\xi).$$

Substituting this in above gives

20 $$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} * \varphi + \int_0^t \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)d\tau$$

The inversion theorem then gives

22 $$u(x,t) = \left(\frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} * \varphi\right)(x) + \int_0^t \left(\frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(x)\,d\tau$$

: pts. to here.

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y,\tau)\,dy\,d\tau}$$

6.(25 pts.) (a) Find a solution to $\overset{①}{u_{xx}+u_{yy}=0}$ for $0<x<1$ and $0<y<1$, subject to the boundary conditions $u(x,0)\overset{②}{=}0\overset{③}{=}u(x,1)$ if $0\le x\le 1$ and $u(0,y)\overset{④}{=}0$, $u(1,y)\overset{⑤}{=}3\sin(\cancel{x})-\sin(\cancel{3x})$ if $0\le y\le 1$.
(b) Show that the solution to the problem in part (a) is unique. πy $3\pi y$

(a) We use separation of variables. We seek nontrivial solutions of ①-②-③-④ of
2 pts. to here. the form $X(x)Y(y)=u(x,y)$. Substituting in ① gives $X''(x)Y(y)+X(x)Y''(y)=0$ for all (x,y) in the unit square. Rearranging $+\dfrac{X''(x)}{X(x)}=-\dfrac{Y''(y)}{Y(y)}=$ constant $=\lambda$. Substituting in ②, ③, and ④ and using the fact that $u(x,y)$ is not identically we see that

$X(x)Y(0)=0$ and $X(x)Y(1)$ for all $0\le x\le 1$ implies $Y(0)=Y(1)=0$ and

$X(0)Y(y)=0$ for all $0\le y\le 1$ implies $X(0)=0$. Therefore we have the coupled system

4 $$\begin{cases} X''(x)-\lambda X(x)=0 \text{ if } 0<x<1 \text{ and } X(0)=0 \\ \boxed{Y''(y)+\lambda Y(y)=0 \text{ if } 0<y<1 \text{ and } Y(0)=0=Y(1).} \end{cases}$$
7 $\leftarrow$ eigenvalue problem

Since the operator $T=\dfrac{-d^2}{dy^2}$ is symmetric on $V_D=\{f\in C^2[0,1]: f(0)=0=f(1)\}$ it follows that the eigenvalues are real. Since $\overline{f(0)}f'(0)-\overline{f(1)}f'(1)=0$ for all $f\in V_D$, it
8 follows that the eigenvalues are all positive.

Case $\lambda>0$, say $\lambda=\beta^2$ where $\beta>0$. We must solve $Y''(y)+\beta^2Y(y)$ if subject to $Y(0)=0=$
The general solution of the DE is $Y(y)=c_1\cos(\beta y)+c_2\sin(\beta y)$. Applying the B.C.'s we see that $0=c_1$ and $0=c_1\cos(\beta)+c_2\sin(\beta)$. Therefore $c_1=0$ and $\beta=\beta_n=n\pi$ $(n=1,2,\dots)$
12 for nontrivial solutions. Hence $\lambda_n=(n\pi)^2$ and $Y_n(y)=\sin(n\pi y)$ are the eigenvalues and eigenfunctions, respectively. $(n=1,2,3,\dots)$.

The solution of the x-equations: $X_n''(x)-(n\pi)^2X_n(x)=0$, $X_n(0)=0$, are (up to
14 a constant factor) $X_n(x)=\sinh(n\pi x)$.

By the superposition principle $u(x,y)=\sum_{n=1}^{\infty}c_nX_n(x)Y_n(y)=\sum_{n=1}^{\infty}c_n\sinh(n\pi x)\sin(n\pi y)$

is a formal solution to ①-②-③-④ for any choice of constants $c_1, c_2, c_3, \dots$

(OVER)

We want to choose the constants so ⑤ is satisfied:

$$3\sin(\pi y) - \sin(3\pi y) = u(1,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi)\sin(n\pi y) \qquad \text{for all } 0 \le y \le 1$$

That is: $3\sin(\pi y) - \sin(3\pi y) = c_1\sinh(\pi)\sin(\pi y) + c_2\sinh(2\pi)\sin(2\pi y) + c_3\sinh(3\pi)\sin(3\pi y) + \dots$ for all $0 \le y \le 1$. By inspection, a solution occurs if we choose

18 $$c_1 = \frac{3}{\sinh(\pi)}, \quad c_3 = \frac{-1}{\sinh(3\pi)}, \quad \text{and} \quad c_n = 0 \text{ for all other } n.$$

Therefore

20 pts.
$$\boxed{u(x,y) = \frac{3\sinh(\pi x)\sin(\pi y)}{\sinh(\pi)} - \frac{\sinh(3\pi x)\sin(3\pi y)}{\sinh(3\pi)}}$$

solves ①-②-③-④-⑤.

5 (b) Suppose that $u = v(x,y)$ is another (continuous) solution to ①-②-③-④-⑤. Then the continuous function $w(x,y) = u(x,y) - v(x,y)$ solves $w_{xx} + w_{yy} = 0$ in the unit square $S = \{(x,y) : 0<x<1, 0<y<1\}$ and $w(x,y) = 0$ if $(x,y) \in \partial S$, the boundary of S. By the maximum/minimum principle for continuous solutions to Laplace's equation

$$0 = \min_{(x,y)\in\partial S} w(x,y) = \min_{(x,y)\in\overline{S}} w(x,y) \le \max_{(x,y)\in\overline{S}} w(x,y) = \max_{(x,y)\in\partial S} w(x,y) = 0.$$

That is, $w(x,y) \equiv 0$ on $\overline{S}$ so $u(x,y) = v(x,y)$ for all $0 \le x \le 1$ and $0 \le y \le 1$.

This shows that there is only one (continuous) solution to ①-②-③-④-⑤.

7.(25 pts:) Consider the constant function $\phi(x)=100$ on the closed interval $[0,1]$.

(a) Show that the Fourier sine series of ϕ on $[0,1]$ is

$$\frac{400}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}.$$

(b) Does the Fourier sine series of ϕ converge to ϕ in the L^2 – sense on $[0,1]$? Justify your answer.

(c) For which x in $[0,1]$ does the Fourier sine series of ϕ converge pointwise to $\phi(x)$? Justify your answer.

(d) Does the Fourier sine series of ϕ converge to ϕ uniformly on $[0,1]$? Justify your answer.

(e) Use the results above to help find the sum of the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$.

(f) Use the results above to help find the sum of the series $\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$. (Hint: Parseval's identity.)

7 (a) $b_n = \dfrac{\langle \varphi, \sin(n\pi\,\cdot)\rangle}{\langle \sin(n\pi\,\cdot), \sin(n\pi\,\cdot)\rangle} = \dfrac{\int_0^1 \varphi(x)\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} = 2\int_0^1 100\sin(n\pi x)\,dx = \dfrac{-200}{n\pi}\cos(n\pi x)\Big|_0^1$

$= \dfrac{-200\left((-1)^n - 1\right)}{n\pi} = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ \dfrac{400}{(2k+1)\pi} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$ Therefore the sine series

for φ on $[0,1]$ is $\sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} b_n \sin(n\pi x) = \boxed{\sum_{k=0}^{\infty} \frac{400\sin((2k+1)\pi x)}{(2k+1)\pi}}$.

3 (b) $\int_0^1 |\varphi(x)|^2\,dx = \int_0^1 10{,}000\,dx = 10{,}000 < \infty$. By Theorem 3, the Fourier sine series for φ $\boxed{\text{converges}}$ in the L^2-sense (or mean-square sense) to φ on $[0,1]$

6 (c) Since $\varphi(x) = 100$ and $\varphi'(x) = 0$ are continuous on $[0,1]$, Theorem 4 shows that the Fourier sine series for φ converges pointwise to $\varphi(x) = 100$ for all x in $(0,1)$. Notice that the Fourier sine series for φ is 0 when $x = 0$ or $x = 1$. Therefore the Fourier sine series does not converge to $\varphi(0) = 100 = \varphi(1)$ at $x=0$ and $x=1$.

Answer: For $\boxed{0 < x < 1}$, the Fourier sine series for φ converges to $\varphi(x)$ pointwise.

3 (d) Since uniform convergence on $[a,b]$ implies pointwise convergence on $[a,b]$, it follows from part (c) that the Fourier sine series for φ does $\boxed{\text{not converge uniformly}}$ to φ on $[0,1]$

(OVER)

3 (e) By parts (a) and (c), $100 = \frac{400}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}$ for all x in $(0,1)$.

k	$\sin((2k+1)\pi/2)$
0	$\sin(\pi/2) = 1$
1	$\sin(3\pi/2) = -1$
2	$\sin(5\pi/2) = 1$
3	$\sin(7\pi/2) = -1$
$\vdots$	$\vdots$

Taking $x = 1/2$ in this identity yields

$$100 = \frac{400}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi/2)}{2k+1}$$

$$\Rightarrow \quad \boxed{\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}}$$

3 (f) By Parseval's identity,

$$\sum_{n=1}^{\infty}|b_n|^2\int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 |\varphi(x)|^2\,dx$$

so

$$\sum_{k=0}^{\infty}\left(\frac{400}{(2k+1)\pi}\right)^2\cdot\frac{1}{2} = 10{,}000$$

$$\Rightarrow \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{2(10{,}000)\pi^2}{400^2}$$

$$\boxed{\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}}$$

8.(25 pts.) Consider a metal rod of length 1 meter, insulated along its sides but not at its ends, which is initially at temperature 100^0 C. Suddenly both ends are inserted in an ice bath of temperature 0^0 C.
(a) Write the partial differential equation, boundary conditions, and initial condition that model the temperature $u = u(x,t)$ of the rod at position x in $[0,1]$ and time $t > 0$.
(b) Find a formula for the temperature $u = u(x,t)$. (Hint: You may find the results of problem ~~6~~ 7 useful.)

10 (a)

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} \overset{①}{=} 0 & \text{if } 0<x<1 \text{ and } 0<t<\infty, \\ u(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t) & \text{if } t \geq 0, \\ u(x,0) \overset{④}{=} 100 & \text{if } 0 \leq x \leq 1. \end{cases}$$

4

2 + 2

2

15 (b) We use separation of variables. We seek nontrivial solutions of ①-②-③ of the form $u(x,t) = X(x)T(t)$. Then ① implies $-\dfrac{T'(t)}{kT(t)} = -\dfrac{X''(x)}{X(x)} = \lambda$.

1 pt. to here

Also ② and ③ imply $X(0) = 0$ and $X(1) = 0$ since $u(x,t) = X(x)T(t)$ is not identically zero. Therefore

$$\begin{cases} \boxed{X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(1)} \leftarrow \text{Eigenvalue problem} \\ T'(t) + \lambda k T(t) = 0. \end{cases}$$

4

By the work in problem 6, the eigenvalues and eigenfunctions are, respectively, $\lambda_n = (n\pi)^2$ and $X_n(x) = \sin(n\pi x)$ $(n = 1, 2, 3, \ldots)$. The solution to the

6

t-equation $T_n'(t) + k(n\pi)^2 T_n(t) = 0$ is $T_n(t) = e^{-kn^2\pi^2 t}$ $(n = 1, 2, 3, \ldots)$,

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at least up to a constant factor. By superposition,

10

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-kn^2\pi^2 t}$$

solves ①-②-③ formally for any choice of constants $c_1, c_2, c_3, \ldots$ We need to choose the constants so condition ④ is satisfied:

11

$$100 = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \quad \text{for all } x \text{ in } (0,1).$$

(OVER)

By problem 7, we should choose

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$$c_n = \begin{cases} 0 & \text{if } n = 2j \text{ is even,} \\ \dfrac{400}{(2j+1)\pi} & \text{if } n = 2j+1 \text{ is odd.} \end{cases}$$

Thus,

15 pts. to here.

$$\boxed{u(x,t) = \sum_{j=0}^{\infty} \frac{400 \sin((2j+1)\pi x)}{\pi(2j+1)} e^{-k(2j+1)^2\pi^2 t}}$$

solves ①-②-③-④.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi-a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \qquad (8)$$

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

Math 325

Summer 2009

Final Exam

n : 16

μ : 141.0

σ : 30.2

Distribution of Scores:

		frequency
174 – 200	A	1
146 – 173	B	8
120 – 145	C(graduate), B(undergraduate)	3
100 – 119	C	1
0 – 99	F(graduate), D(undergraduate)	3