Mathematics 325 | Exam II Summer 2014 | Name: Dr. Grow (1 pt.)

1.(33 pts.) Solve $u_t - u_{xx} = 0$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = x^3$ if $-\infty < x < \infty$. You may find the following identities useful:

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}, \quad \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp = 0 = \int_{-\infty}^{\infty} p e^{-p^2}\,dp.$$

We know that if $\varphi = \varphi(x)$ is a <u>bounded</u> continuous function on $(-\infty,\infty)$ then $u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy$ is a solution to $u_t - ku_{xx} = 0$ in the upper half-plane and satisfies $u(x,0) = \varphi(x)$ if $-\infty < x < \infty$. In our case, $k=1$ and $\varphi(x) = x^3$ is continuous but <u>unbounded</u> on $(-\infty,\infty)$. We hope to obtain a solution to our problem using the formula for $u(x,t)$ above but we will need to check our answer because the conditions under which the solution is valid are not met.

6 pts. to here.

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^3\,dy.$$ Let $p = \frac{y-x}{\sqrt{4t}}$. Then $dp = \frac{dy}{\sqrt{4t}}$ so

12 pts. to here. 17 pts. to here.

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}(p\sqrt{4t}+x)^3\,dp = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\left(p^3 4t\sqrt{4t} + 3p^2 4tx + 3p\sqrt{4t}x^2 + x^3\right)dp$$

23 pts. to here

$$= \frac{4t\sqrt{4t}}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp}_{0} + \frac{12tx}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp}_{\frac{\sqrt{\pi}}{2}} + \frac{3\sqrt{4t}x^2}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} p e^{-p^2}\,dp}_{0} + \frac{x^3}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-p^2}\,dp}_{\sqrt{\pi}}$$

29 pts. to here.

$$= \boxed{x^3 + 6tx}.$$

33 pts. to here.

**Check:** $u_t - u_{xx} = 6x - 6x \overset{\checkmark}{=} 0$

$u(x,0) \overset{\checkmark}{=} x^3$

2.(33 pts.) Use Fourier transform methods to find a formula for the solution to $u_t - ku_{xx} = f(x,t)$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = 0$ if $-\infty < x < \infty$.

Suppose $u = u(x,t)$ is a solution to the problem. Then

$$(*) \qquad u_t(x,t) - k u_{xx}(x,t) = f(x,t)$$

for all $(x,t)$ in the upper half-plane. Taking the Fourier transform of $(*)$ with respect to $x$ holding $t$ fixed gives

2 pts. to here

$$\mathcal{F}\big(u_t(\cdot,t) - ku_{xx}(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

for all $-\infty < \xi < \infty$ so

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$$\frac{\partial}{\partial t}\mathcal{F}\big(u(\cdot,t)\big)(\xi) - k(i\xi)^2\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

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$$(**) \qquad \frac{\partial \mathcal{F}}{\partial t}\big(u(\cdot,t)\big)(\xi) + k\xi^2\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

This is a first order linear ODE in the independent variable $t$ (with parameter $\xi$) so an integrating factor is $\mu(t) = e^{\int k\xi^2\,dt} = e^{k\xi^2 t}$.

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Multiplying through $(**)$ by the integrating factor and simplifying yields

12 pts. to here

$$\frac{\partial}{\partial t}\Big[e^{k\xi^2 t}\mathcal{F}\big(u(\cdot,t)\big)(\xi)\Big] = e^{k\xi^2 t}\frac{\partial \mathcal{F}}{\partial t}\big(u(\cdot,t)\big)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}\big(u(\cdot,t)\big)(\xi) = e^{k\xi^2 t}\mathcal{F}\big(f(\cdot,t)\big)(\xi)$$

and integrating with respect to $t$ holding $\xi$ fixed,

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$$e^{k\xi^2 t}\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \int_0^t e^{k\xi^2\tau}\mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,d\tau + c(\xi).$$

Evaluating at $t=0$ and using the initial condition gives

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$$0 = \mathcal{F}(0)(\xi) = \mathcal{F}\big(u(\cdot,0)\big)(\xi) = c(\xi) \qquad (-\infty < \xi < \infty).$$

Therefore

$$\mathcal{F}\big(u(\cdot,t)\big)(\xi) = e^{-k\xi^2 t}\int_0^t e^{k\xi^2\tau}\mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,d\tau$$

19 pts. to here

$$(***) \qquad = \int_0^t e^{-k\xi^2(t-\tau)}\mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\,d\tau.$$

Applying formula I in the table of Fourier transforms with $a = \dfrac{1}{4k(t-\tau)}$ gives $\mathcal{F}\left(\sqrt{2a}\, e^{-a(\cdot)^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}}$ so

21 pts. to here $$\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{1}{4k(t-\tau)}(\cdot)^2}\right)(\xi) = e^{-k\xi^2(t-\tau)}.$$

Substituting in (***) and using the convolution property $\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\,\mathcal{F}(g)(\xi)$ leads to

23 $$\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t \mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\right)(\xi)\,\mathcal{F}(f(\cdot,\tau))(\xi)\,d\tau$$

26 $$= \int_0^t \frac{1}{\sqrt{2\pi}}\,\mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(\xi)\,d\tau.$$

Interchanging the order of integration with respect to $\tau$ and $x$ (in the Fourier transform) produces

28 pts. to here $$\mathcal{F}(u(\cdot,t))(\xi) = \mathcal{F}\left(\int_0^t \frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{\frac{-(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\,d\tau\right)(\xi).$$

By the Fourier inversion formula and smoothness of the functions involved, we have

31 $$u(x,t) = \int_0^t \left(\frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}} * f(\cdot,\tau)\right)(x)\,d\tau$$

33 pts. to here $$= \boxed{\int_0^t \int_0^\infty \frac{1}{\sqrt{4k\pi(t-\tau)}}\, e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y,\tau)\,dy\,d\tau}.$$

3.(33 pts.) Solve $u_{xx} + u_{yy} \overset{(1)}{=} 0$ in the square $-1 < x < 1$, $-1 < y < 1$, subject to the boundary conditions $u(-1, y) \overset{(2)}{=} 0$ if $-1 \le y \le 1$, $u(x, -1) \overset{(3)}{=} 0 \overset{(4)}{=} u(x, 1)$ if $-1 \le x \le 1$, and $u(1, y) \overset{(5)}{=} \cos\left(\frac{\pi}{2} y\right) + \sin(\pi y)$ if $-1 \le y \le 1$. You may find useful the fact that if $\alpha$ is a positive constant then every solution to

$$X''(x) - \alpha^2 X(x) = 0,\ X(-1) = 0$$

is a constant multiple of $X(x) = \sinh(\alpha(x+1))$.

We use the method of separation of variables since the region of solution is bounded. We seek nontrivial solutions of the form $u(x,y) = X(x)Y(y)$ to the homogeneous portion of the problem: (1) - (2) - (3) - (4). Then (1) implies

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \text{so} \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{constant} = \lambda.$$

(2) implies $X(-1)Y(y) = 0$ for all $-1 \le y \le 1$ and (3) - (4) imply $X(x)Y(-1) = 0 = X(x)Y(1)$ for all $-1 \le x \le 1$. Since we seek nontrivial solutions, it follows that

$$X''(x) - \lambda X(x) \overset{(6)}{=} 0,\quad X(-1) \overset{(7)}{=} 0$$

$$\boxed{Y''(y) + \lambda Y(y) \overset{(8)}{=} 0,\quad Y(-1) \overset{(9)}{=} 0 \overset{(10)}{=} Y(1).} \leftarrow \text{Eigenvalue problem.}$$

We may assume the eigenvalues of (8) - (9) - (10) are real numbers.

<u>Case $\lambda > 0$ (say $\lambda = \alpha^2$ for some $\alpha > 0$)</u>: Then (8) becomes $Y''(y) + \alpha^2 Y(y) = 0$.

The general solution is $Y(y) = c_1 \cos(\alpha y) + c_2 \sin(\alpha y)$. Applying (9) - (10) gives

$$0 = Y(1) = c_1 \cos(\alpha) + c_2 \sin(\alpha)$$

$$0 = Y(-1) = c_1 \cos(-\alpha) + c_2 \sin(-\alpha) = c_1 \cos(\alpha) - c_2 \sin(\alpha).$$

Adding equations produces $0 = 2c_1 \cos(\alpha)$ and subtracting equations leads to $0 = 2c_2 \sin(\alpha)$. For a nontrivial solution to exist we must have either $\cos(\alpha) = 0$ or $\sin(\alpha)$. In other words $\alpha \in \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \ldots \right\}$, so $\alpha_k = k\frac{\pi}{2}$ where $k = 1, 2, 3, \ldots$ The eigenvalues are $\lambda_k = \left(k\frac{\pi}{2}\right)^2$ $(k = 1, 2, 3, \ldots)$ and the corresponding eigenfunctions are $Y_{2n-1}(y) = \cos\left((2n-1)\frac{\pi}{2} y\right)$ and

$\uparrow$ (k odd)

$\underset{\substack{\uparrow \\ (k \text{ even})}}{Y_{2n}}(y) = \sin(n\pi y) \qquad (n = 1, 2, 3, \dots).$

Case $\lambda = 0$: Then ⑧ becomes $Y''(y) = 0$ so $Y(y) = c_1 y + c_2$. Applying ⑨-⑩ we have $-c_1 + c_2 = 0 = c_1 + c_2$ so $c_1 = c_2 = 0$. Therefore 0 is not an eigenvalue.

Case $\lambda < 0$ (say $\lambda = -\alpha^2$ for some $\alpha > 0$): Then ⑧ becomes $Y''(y) - \alpha^2 Y(y) = 0$ with general solution $Y(y) = c_1 \sinh(\alpha y) + c_2 \cosh(\alpha y)$. Applying ⑨-⑩ yields

$$0 = Y(1) = c_1 \sinh(\alpha) + c_2 \cosh(\alpha)$$

$$0 = Y(-1) = c_1 \sinh(-\alpha) + c_2 \cosh(-\alpha) = -c_1 \sinh(\alpha) + c_2 \cosh(\alpha).$$

Adding equations gives $0 = 2c_2 \cosh(\alpha) \Rightarrow c_2 = 0$. Subtracting equations gives $0 = 2c_1 \sinh(\alpha) \Rightarrow c_1 = 0$. Therefore there are no negative eigenvalues.

We need to solve ⑥-⑦ if $\lambda$ is an eigenvalue, i.e. $\lambda_k = \left(\frac{k\pi}{2}\right)^2$ $(k = 1, 2, 3, \dots)$

⑥-⑦ becomes $X_k''(x) - \left(\frac{k\pi}{2}\right)^2 X_k(x) = 0$, $X_k(-1) = 0$, so by the hint $X_k(x) = \sinh\left(\frac{k\pi}{2}(x+1)\right)$. Then

$$u_k(x,y) = X_k(x) Y_k(y) = \begin{cases} \sinh\left(\frac{(2n-1)\pi}{2}(x+1)\right)\cos\left(\frac{(2n-1)\pi}{2} y\right) & \text{if } k = 2n-1 \text{ is odd} \\ \sinh\left(n\pi(x+1)\right)\sin(n\pi y) & \text{if } k = 2n \text{ is even} \end{cases}$$

By the superposition principle

$$u(x,y) = \sum_{n=1}^{N} \left[ a_{2n-1} \sinh\left(\frac{(2n-1)\pi}{2}(x+1)\right)\cos\left(\frac{(2n-1)\pi}{2} y\right) + a_{2n} \sinh(n\pi(x+1))\sin(n\pi y) \right]$$

solves ①-②-③-④ for any integer $N \geq 1$ and any choice of constants $a_1, a_2, \dots, a_{2N}$.

We want condition ⑤ to be satisfied as well: If $-1 \leq y \leq 1$ then

$$\cos\left(\frac{\pi y}{2}\right) + \sin(\pi y) = u(1,y) = a_1 \sinh(\pi)\cos\left(\frac{\pi y}{2}\right) + a_2 \sinh(2\pi)\sin(\pi y) + a_3 \sinh(3\pi)\cos\left(\frac{3\pi y}{2}\right) + \dots$$

Clearly, we may choose $N = 1$ and $a_1 = \frac{1}{\sinh(\pi)}$, $a_2 = \frac{1}{\sinh(2\pi)}$. That is,

$$\boxed{u(x,y) = \frac{\sinh\left(\frac{\pi}{2}(x+1)\right)\cos\left(\frac{\pi}{2} y\right)}{\sinh(\pi)} + \frac{\sinh(\pi(x+1))\sin(\pi y)}{\sinh(2\pi)}}$$ solves ①-②-③-④-⑤.

# A Brief Table of Fourier Transforms

| | $f(x)$ | $\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$ |
|---|---|---|
| A. | $\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$ | $\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$ |
| B. | $\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$ | $\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$ |
| C. | $\frac{1}{x^2+a^2}\quad (a>0)$ | $\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$ |
| D. | $\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$ | $\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$ |
| E. | $\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$ | $\frac{1}{(a+i\xi)\sqrt{2\pi}}$ |
| F. | $\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$ | $\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$ |
| G. | $\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$ | $\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$ |
| H. | $\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$ | $\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$ |
| I. | $e^{-ax^2}\quad (a>0)$ | $\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$ |
| J. | $\frac{\sin(ax)}{x}\quad (a>0)$ | $\begin{cases}0 & \text{if } \lvert\xi\rvert\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a.\end{cases}$ |

Math 325

Exam II

Summer 2014

$n = 16$

$\mu = 63.9$

$\sigma = 20.6$

| Range | Grad. Letter Grade | Undergrad. Letter Grade | Frequency |
|---|---|---|---|
| 87–100 | A | A | 2 |
| 73–86 | B | B | 5 |
| 60–72 | C | B | 3 |
| 50–59 | C | C | 2 |
| 0–49 | F | D | 4 |