1. (25 pts.) Consider the vector space $V = \{ \varphi \in C^2(0,1) : \varphi(1) = 0, \varphi' \text{ and } \varphi'' \text{ bounded on } (0,1) \}$, equipped with the inner product

$$<f, g> = \frac{1}{0} f(x)g(x) dx.$$ 

(a) Show that the operator $T$ defined on $V$ by

$$Tf(x) = \frac{1}{x} \frac{d}{dx} \left( x \frac{df}{dx} - \frac{\mu^2}{x^2} f(x) \right)$$

is symmetric.

(b) Are the eigenvalues of $T$ on $V$ real? Why or why not?

(c) Are eigenfunctions corresponding to distinct eigenvalues of $T$ on $V$ orthogonal? Why or why not?

(d) Are the eigenvalues of $T$ on $V$ nonnegative? Why or why not?

(a) Let $f$ and $g$ belong to $V$. Then

$$<Tf, g> = \int_0^1 \left[ \frac{d}{dx} \left( x \frac{df}{dx} - \frac{\mu^2}{x^2} f(x) \right) \right] g(x) dx$$

$$= \int_0^1 \frac{d}{dx} \left( x \frac{df}{dx} \right) g(x) dx - \int_0^1 \frac{\mu^2}{x^2} f(x) g(x) dx.$$  

Using integration by parts twice on the first term in the right member of (1) gives

$$\int_0^1 \left( x f'(x) \right) g(x) dx = \left. \left( x f(x) g(x) \right) \right|_0^1 - \int_0^1 x f'(x) g(x) dx$$

$$= \left[ x f'(x) g(x) - x g(x) f'(x) \right]_0^1 + \int_0^1 f(x) \left( x g'(x) \right) dx.$$  

Since $f(x)g(x)$ and $g(x)f(x)$ are bounded for $x$ in (0,1), the squeeze theorem implies $\lim_{x \to 0^+} \left[ x f(x) g(x) \right] = 0$. Similarly, since $g(1) = 0 = f(1)$,

$$\lim_{x \to 1^-} \left[ x f(x) g(x) \right] = 0$$

and hence $\left[ x f'(x) g(x) - x g(x) f'(x) \right]_0^1 = 0$. Thus (1) becomes

$$\int_0^1 \left( x f'(x) \right) g(x) dx = \int_0^1 f(x) \left( x g'(x) \right) dx.$$ 

Substituting this in (1) gives

$$<Tf, g> = \int_0^1 f(x) \left( x g'(x) \right) dx - \int_0^1 \frac{\mu^2}{x^2} f(x) g(x) dx = \int_0^1 f(x) \left[ \frac{1}{x} \left( x g'(x) \right) - \frac{\mu^2}{x^2} g(x) \right] dx = <f, Tg>.$$ 

I.e., $T$ is symmetric on $V$. 

(b) Yes, the eigenvalues of $T$ on $V$ are real numbers since $T$ is symmetric on $V$. This is a consequence of Theorem 2 in the lecture notes for Section 5.3.

(c) Yes, eigenfunctions corresponding to distinct eigenvalues of $T$ on $V$ are orthogonal since $T$ is symmetric on $V$. This is a consequence of Theorem 1 in the lecture notes for Section 5.3.

\[ \text{(if } \mu^2 \neq 0 \text{)} \]

(d) No, the eigenvalues of $T$ on $V$ are not negative. To see this, let $\lambda$ be an eigenvalue of $T$ on $V$ and let $f$ be a function in $V$ that is not identically zero and which satisfies $Tf = \lambda f$. Then $\lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle Tf, f \rangle$.

Applying (*) and the first line of (**) with $g = f$ then gives

\[
\lambda \langle f, f \rangle = \int_0^1 \left[ \frac{1}{x} \left( \frac{d}{dx} (x f(x)) \right)^2 - \frac{\mu^2}{x} f(x)^2 \right] f(x) \, dx
= \left. x f'(x) f(x) \right|_0^1 - \int_0^1 x f'(x) f(x) \, dx - \mu^2 \int_0^1 \frac{1}{x} f(x)^2 \, dx
= -\int_0^1 f'(x)^2 \, dx - \mu^2 \int_0^1 \frac{1}{x} f(x)^2 \, dx
= 0 \quad \text{(positive since $f$ is continuous and not identically zero on $[0,1]$)}
\]

Since $\langle f, f \rangle > 0$, it follows that $\lambda < 0$ (if $\mu^2 \neq 0$).
2. (35 pts.) Consider the constant function \( \varphi(x) = 100 \) on the closed interval \([0,1]\).

(a) Show that the Fourier sine series of \( \varphi \) on \([0,1]\) is
\[
\frac{400}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}.
\]

(b) Does the Fourier sine series of \( \varphi \) converge to \( \varphi \) in the \( L^2 \)-sense on \([0,1]\)? Justify your answer.

(c) For which \( x \) in \([0,1]\) does the Fourier sine series of \( \varphi \) converge pointwise to \( \varphi(x) \)? Justify your answer.

(d) Does the Fourier sine series of \( \varphi \) converge to \( \varphi \) uniformly on \([0,1]\)? Justify your answer.

(e) Use the results above to help find the sum of the series \( \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \).

(f) Use the results above to help find the sum of the series \( \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \). (Hint: Parseval’s identity.)

(a) The Fourier sine series of \( \varphi \) on \([0,1]\) is
\[
\varphi(x) \sim \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{1} \right)
\]

where
\[
b_n = \frac{\langle \varphi, \sin(n\pi x) \rangle}{\langle \sin(n\pi x), \sin(n\pi x) \rangle} = \frac{1}{\pi} \left[ \frac{200}{n\pi} \cos(n\pi) \right]_0^1 = \frac{200(1-(-1)^n)}{n\pi} = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ \frac{400}{n\pi(2k+1)} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}
\]

Therefore \( \varphi(x) \sim \sum_{k=0}^{\infty} \frac{400}{\pi(2k+1)} \sin((2k+1)\pi x) \) is the Fourier sine series of \( \varphi(x) = 100 \) on \([0,1]\).

(c) Since \( \varphi(x) = 100 \) is a continuous function on \([0,1]\) and \( \varphi'(x) = 0 \) is continuous (and hence piecewise continuous) on \([0,1]\), Convergence Theorem 4(i) guarantees that the Fourier sine series of \( \varphi \) converges to \( \varphi(x) = 100 \) for all \( x \) in the open interval \( 0 < x < 1 \). Note however that the Fourier series of \( \varphi \) at the endpoints \( x = 0 \) and \( x = 1 \) do not converge to \( \varphi(0) = 100 \) and \( \varphi(1) = 100 \), respectively:

\[
\sum_{k=0}^{\infty} \frac{400}{\pi(2k+1)} \sin((2k+1)\pi x) = 0 \text{ at } x = 0, \quad \sum_{k=0}^{\infty} \frac{400}{\pi(2k+1)} \sin((2k+1)\pi x) = 0 \text{ at } x = 1.
\]

Therefore the Fourier sine series of \( \varphi \) converges to \( \varphi(x) \) precisely when \( 0 < x < 1 \).

(b) \( \int_0^1 |\varphi(x)|^2 dx = \int_0^1 100^2 dx = 10,000 < \infty \) so Convergence Theorem 3 guarantees...
that the Fourier sine series of \( \varphi \) converges to \( \varphi \) in the \( L^2 \)-sense on \([0,1]\):

\[
\lim_{N \to \infty} \int_0^1 \left| \varphi(x) - \sum_{k=0}^N \frac{400}{\pi (2k+1)} \sin((2k+1)\pi x) \right|^2 dx = 0.
\]

(d) [No] the Fourier sine series of \( \varphi \) does not converge uniformly to \( \varphi \) on \([0,1]\). To see this, note that part (c) implies

\[
S(\varphi; x) = \lim_{N \to \infty} \sum_{k=0}^N \frac{400}{\pi (2k+1)} \sin((2k+1)\pi x) = \begin{cases} 100 & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0 \text{ or } x = 1. \end{cases}
\]

If the convergence of the Fourier sine series of \( \varphi \) were uniform on \([0,1]\) then the sum function \( S(\varphi; \cdot) \) would be continuous on \([0,1]\). But \( S(\varphi; \cdot) \) is clearly discontinuous at \( x = 0 \) and \( x = 1 \) so the convergence cannot be uniform.

(e) Applying identity (a) when \( x = \frac{1}{2} \) yields

\[
\frac{400}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \sum_{k=0}^{\infty} \frac{400}{\pi (2k+1)} \sin((2k+1)\pi \frac{1}{2}) = 100,
\]

and it follows that \( \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} \).

(f) Let \( \{\xi_n\}_{n=1}^{\infty} \) be a complete orthogonal set on \((a,b)\). If \( f \in L^2(a,b) \) has Fourier series \( \sum_{n=1}^{\infty} A_n \xi_n(x) \) with respect to \( \{\xi_n\}_{n=1}^{\infty} \) on \((a,b)\), then Parseval's identity holds:

\[
\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |\xi_n(x)|^2 dx = \int_a^b |f(x)|^2 dx.
\]

Apply this with \( \{\xi_n(x)\}_{n=1}^{\infty} = \{\sin(n\pi x)\}_{n=1}^{\infty} \) and \( f(x) = 100 \) on \((0,1)\) to get

\[
\frac{80000}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{k=0}^{\infty} \left( \frac{400}{\pi (2k+1)} \right)^2 \int_0^1 \sin^2((2k+1)\pi x) dx = \int_0^1 (100\sin x)^2 dx = 10,000,
\]

and it follows that

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.
\]
3. (40 pts.) Consider a metal rod of length 1 meter, insulated along its sides but not at its ends, which is initially at temperature \(100^\circ\text{C}\). Suddenly both ends are inserted in an ice bath of temperature \(0^\circ\text{C}\).

(a) Write the partial differential equation and initial/boundary conditions that model the temperature \(u = u(x,t)\) of the rod at position \(x\) in \([0,1]\) and time \(t \geq 0\).

(b) Find a formula for the temperature \(u = u(x,t)\). (Hint: You may find the results of problem 2 useful.)

(c) Show that the solution to the problem in part (a) is unique. (Hint: Why doesn’t the maximum/minimum principle apply?)

\[ \begin{align*}
\begin{cases}
    u_t - ku_{xx} = 0 & \text{if } 0 < x < 1, 0 < t < \infty, \\
    u(0,t) = 0 = u(1,t) & \text{if } 0 \leq t < \infty, \\
    u(x,0) = 100 & \text{if } 0 < x < 1.
\end{cases}
\end{align*} \]

2.0 (b) We use the method of separation of variables to solve \(1 \rightarrow 2 \rightarrow 3 \rightarrow 4\). We seek nontrivial solutions of \(1 \rightarrow 5 \rightarrow 6 \) of the form \(u(x,t) = \xi(x)T(t)\). Substituting in \(1\) gives

\[ \xi(x)T'(t) - k\xi''(x)T(t) = 0 \]

and rearranging gives \(\frac{\xi''(x)}{\xi(x)} = \frac{-T'(t)}{kT(t)} = \text{constant} = \lambda\). Substituting in \(2\) and \(3\) gives \(\xi(0)T(t) = 0 = \xi(1)T(t)\) if \(0 < t < \infty\). Since \(u(x,t) = \xi(x)T(t)\) is not identically zero on \(0 < x < 1, 0 < t < \infty\), it follows that \(\xi(0) = 0 = \xi(1)\). Hence we have the coupled system of ODEs and BCs:

\[ \begin{align*}
\xi''(x) + \lambda \xi(x) &= 0, & \xi(0) = \xi(1) \\
T'(t) + \lambda kT(t) &= 0.
\end{align*} \]

By Section 4.1, the eigenvalue problem \(5 \rightarrow 6 \rightarrow 7\) has eigenvalues \(\lambda_n = (n\pi)^2\) and eigenfunctions \(\xi_n(x) = \sin(n\pi x)\) \((n = 1, 2, 3, \ldots)\). When \(\lambda = \lambda_n = (n\pi)^2\) in \(8\), \(T_n(t) = e^{-k(n\pi)^2 t}\) (up to a multiplicative constant). Therefore \(u_n(x,t) = \xi_n(x)T_n(t) = \sin(n\pi x)e^{-k(n\pi)^2 t}\) solves \(1 \rightarrow 2 \rightarrow 3\) for \(n = 1, 2, 3, \ldots\). The superposition principle then yields a formal solution

\[ u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)e^{-k(n\pi)^2 t} \]

to \(1 \rightarrow 2 \rightarrow 3\) where \(b_1, b_2, b_3, \ldots\) are arbitrary constants. To satisfy \(4\) we need

\[ 100 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{for } 0 < x < 1. \]
Applying problem 2, we see that we should choose the $b_n$'s to be the Fourier sine coefficients of $u(x) = 100$ on $0 < x < 1$; i.e.

$$b_n = \begin{cases} 
0 & \text{if } n = 2j \text{ is even,} \\
\frac{400}{\pi(2j+1)} & \text{if } n = 2j+1 \text{ is odd.}
\end{cases}$$

Therefore

$$u(x,t) = \sum_{j=0}^{\infty} \frac{400}{\pi(2j+1)} \sin((2j+1)\pi x) e^{-k(2j+1)^2 \pi^2 t}$$

solves (1)-(2)-(3)-(4). (c) Let $u = v(x,t)$ denote the solution to (1)-(2)-(3)-(4) obtained in part (b) of this problem and let $u = u_x(x,t)$ denote another solution. Then $v(x,t) = u(x,t) - u_x(x,t)$ is a continuous function in $0 < x < 1, 0 < t < \infty$, which solves

$$\begin{cases} 
v_t - kv_{xx} = 0 & \text{if } 0 < x < 1, 0 < t < \infty, \\
v(0,t) = 0 & v(1,t) \text{ if } 0 \leq t < \infty, \\
v(x,0) = 0 & \text{if } 0 < x < 1.
\end{cases}$$

Let $\Theta(t) = \left( \int_0^1 (v(x,t))^2 \, dx \right)^{1/2}$ be the energy of $v$ at time $t \geq 0$. Then squaring and differentiating with respect to $t$ yields

$$2\Theta(t)\dot{\Theta}(t) = \frac{d}{dt} \int_0^1 (v(x,t))^2 \, dx = \int_0^1 2v(x,t)v_t(x,t) \, dx.$$

Substituting from (4) and integrating by parts gives

$$\Theta(t)\dot{\Theta}(t) = k \int_0^1 v(x,t)v_{xx}(x,t) \, dx = k v(x,t)v_x(x,t) \bigg|_{x=1} - k \int_0^1 v_x(x,t)^2 \, dx.$$

Applying (10)-(11) we see that $k v(x,t)v_x(x,t) \bigg|_{x=0} = 0$, so

$$\Theta(t)\dot{\Theta}(t) = -k \int_0^1 v_x^2(x,t) \, dx \leq 0.$$ 

Since $\Theta(t) \geq 0$, it follows that $\Theta(t) \leq 0$, i.e. $\Theta$ is decreasing on $0 \leq t < \infty$, so by (12) $0 \leq \Theta(t) \leq \Theta(0) = \left( \int_0^1 v(x,0)^2 \, dx \right)^{1/2} = 0$. But the vanishing theorem then yields $v(x,t) = 0$ for all $0 < x < 1, 0 < t < \infty$. This together with (10)-(11)-(12) imply $v(x,t) = 0$ if $0 \leq x \leq 1, 0 \leq t < \infty$. That is, $u_0(x,t) = u_x(x,t)$ if $0 \leq x \leq 1, 0 \leq t < \infty$, so the solution obtained in part (b) of this problem is unique.
Convergence Theorems

Consider the eigenvalue problem
\begin{align*}
(1) & \quad X''(x) + \lambda X(x) = 0 \quad \text{in} \quad a < x < b \\
(2) & \quad \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0 \\
(3) & \quad \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0
\end{align*}
with any symmetric boundary conditions

and let \( \Phi = \{X_1, X_2, X_3, \ldots\} \) be the complete orthogonal set of eigenfunctions for (1)-(2)-(3). Let \( f \) be any absolutely integrable function defined on \( a \leq x \leq b \). Consider the Fourier series for \( f \) with respect to \( \Phi \):

\[ f(x) \approx \sum_{n=1}^{\infty} A_n X_n(x) \]

where

\[ A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \quad (n = 1, 2, 3, \ldots). \]

**Theorem 2.** (Uniform Convergence) If

(i) \( f(x), f'(x), \) and \( f''(x) \) exist and are continuous for \( a \leq x \leq b \) and

(ii) \( f \) satisfies the given symmetric boundary conditions,

then the Fourier series of \( f \) converges uniformly to \( f \) on \( [a, b] \).

**Theorem 3.** (\( L^2 \) – Convergence) If

\[ \int_a^b |f(x)|^2 \, dx < \infty \]

then the Fourier series of \( f \) converges to \( f \) in the mean-square sense in \( (a, b) \).

**Theorem 4.** (Pointwise Convergence of Classical Fourier Series)

(i) If \( f \) is a continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) at \( x \) converges pointwise to \( f(x) \) in the open interval \( a < x < b \).

(ii) If \( f \) is a piecewise continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) converges pointwise at every point \( x \) in \( (-\infty, \infty) \). The sum of the Fourier series is

\[ \sum_{n=1}^{m} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2} \]

for all \( x \) in the open interval \( (a, b) \).

**Theorem 4.** If \( f \) is a function of period \( 2l \) on the real line for which \( f \) and \( f' \) are piecewise continuous, then the classical full Fourier series converges to \( \frac{f(x^+) + f(x^-)}{2} \) for every real \( x \).
number of exams: 8
mean: 63.4
median: 68
standard deviation: 15.5

Distribution of Scores:

<table>
<thead>
<tr>
<th>Range</th>
<th>Graduate Letter Grade</th>
<th>Undergraduate Letter Grade</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>87 - 110</td>
<td>A</td>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>73 - 86</td>
<td>B</td>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>60 - 72</td>
<td>C</td>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>50 - 59</td>
<td>C</td>
<td>C</td>
<td>0</td>
</tr>
<tr>
<td>0 - 49</td>
<td>F</td>
<td>D</td>
<td>3</td>
</tr>
</tbody>
</table>