Mathematics 325 | Exam III, Summer 2013 | Name: Dr. Grow

1.(25 pts.) Consider the vector space $V = \{\varphi \in C^2(0,1] : \varphi(1) = 0,\ \varphi \text{ and } \varphi' \text{ bounded on } (0,1]\}$, equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}x\,dx.$$

(a) Show that the operator T defined on V by

$$Tf(x) = \frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right) - \frac{\mu^2}{x^2}f(x) \qquad (0 < x \le 1)$$

is symmetric.

(b) Are the eigenvalues of T on V real? Why or why not?

(c) Are eigenfunctions corresponding to distinct eigenvalues of T on V orthogonal? Why or why not?

(d) Are the eigenvalues of T on V nonnegative? Why or why not?

10 (a) Let f and g belong to V. Then

$$\langle Tf, g\rangle = \int_0^1 \left[\left(\frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right) - \frac{\mu^2}{x^2}f(x)\right)\overline{g(x)}\right]x\,dx$$

$$= \int_0^1 \frac{d}{dx}\left(x\frac{df}{dx}\right)\overline{g(x)}\,dx - \int_0^1 \frac{\mu^2}{x^2}f(x)\overline{g(x)}\,x\,dx. \qquad (*)$$

Using integration by parts twice on the first term in the right member of $(*)$ gives

$$\int_0^1 (xf'(x))'\overline{g(x)}\,dx = xf'(x)\overline{g(x)}\Big|_0^1 - \int_0^1 xf'(x)\overline{g'(x)}\,dx$$

$$= \left[xf'(x)\overline{g(x)} - x\overline{g'(x)}f(x)\right]\Big|_0^1 + \int_0^1 f(x)\overline{(xg'(x))'}\,dx. \qquad (**)$$

Since $f'(\varepsilon)\overline{g(\varepsilon)}$ and $\overline{g'(\varepsilon)}f(\varepsilon)$ are bounded for ε in $(0,1]$, the squeeze theorem implies $\lim_{\varepsilon\to 0^+}\left[\varepsilon f'(\varepsilon)\overline{g(\varepsilon)} - \varepsilon\overline{g'(\varepsilon)}f(\varepsilon)\right] = 0$. Similarly, since $g(1) = 0 = f(1)$, $\lim_{x\to 1^-}\left[xf'(x)\overline{g(x)} - x\overline{g'(x)}f(x)\right] = 0$ and hence $\left[xf'(x)\overline{g(x)} - x\overline{g'(x)}f(x)\right]\Big|_0^1 = 0$. Thus $(**)$ becomes $\int_0^1 (xf'(x))'\overline{g(x)}\,dx = \int_0^1 f(x)\overline{(xg'(x))'}\,dx$. Substituting this in $(*)$ gives

$$\langle Tf, g\rangle = \int_0^1 f(x)\overline{(xg'(x))'}\,dx - \int_0^1 \frac{\mu^2}{x^2}f(x)\overline{g(x)}\,x\,dx = \int_0^1 f(x)\overline{\left[\frac{1}{x}(xg'(x))' - \frac{\mu^2}{x^2}g(x)\right]}\,x\,dx = \langle f, Tg\rangle.$$

I.e. T is symmetric on V.

5 (b) Yes, the eigenvalues of T on V are real numbers since T is symmetric on V. This is a consequence of Theorem 2 in the lecture notes for Section 5.3.

5 (c) Yes, eigenfunctions corresponding to distinct eigenvalues of T on V are orthogonal since T is symmetric on V. This is a consequence of Theorem 1 in the lecture notes for Section 5.3.

5 (d) No, the eigenvalues of T on V are negative (if $\mu^2 \neq 0$). To see this, let λ be an eigenvalue of T on V and let f be function in V that is not identically zero and which satisfies $Tf = \lambda f$. Then $\lambda\langle f,f\rangle = \langle \lambda f, f\rangle = \langle Tf, f\rangle$. Applying (*) and the first line of (*) with $g = f$ then gives

$$\begin{aligned}
\lambda\langle f,f\rangle &= \int_0^1 \left[\frac{1}{x}(xf'(x))' - \frac{\mu^2}{x^2}f(x)\right]\overline{f(x)}\, x\,dx \\
&= xf'(x)\overline{f(x)}\Big|_0^1 - \int_0^1 x f'(x)\overline{f'(x)}\,dx - \mu^2\int_0^1 \frac{1}{x} f(x)\overline{f(x)}\,dx \\
&= -\int_0^1 |f'(x)|^2 x\,dx - \mu^2 \underbrace{\int_0^1 |f(x)|^2 \frac{1}{x}\,dx}_{\text{(positive since } f \text{ is continuous and not identically zero on } (0,1])} \\
&< 0 \quad (\text{if } \mu^2 \neq 0).
\end{aligned}$$

Since $\langle f,f\rangle > 0$, it follows that $\lambda < 0$ (if $\mu^2 \neq 0$).

2.(35 pts.) Consider the constant function $\varphi(x) = 100$ on the closed interval $[0,1]$.

(a) Show that the Fourier sine series of φ on $[0,1]$ is

$$\frac{400}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}.$$

(b) Does the Fourier sine series of φ converge to φ in the L^2 – sense on $[0,1]$? Justify your answer.

(c) For which x in $[0,1]$ does the Fourier sine series of φ converge pointwise to $\varphi(x)$? Justify your answer.

(d) Does the Fourier sine series of φ converge to φ uniformly on $[0,1]$? Justify your answer.

(e) Use the results above to help find the sum of the series $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$.

(f) Use the results above to help find the sum of the series $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$. (Hint: Parseval's identity.)

10 (a) The Fourier sine series of φ on $[0,\ell]$ is $\varphi(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$ where

$$b_n = \frac{\langle \varphi, \sin(n\pi\cdot/\ell)\rangle}{\langle \sin(n\pi\cdot/\ell), \sin(n\pi\cdot/\ell)\rangle} = \frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$ for $n = 1,2,3,\ldots$. In our case

$\ell = 1$ and $\varphi(x) = 100$ if $0 \le x \le 1$. Consequently,

$$b_n = 2\int_0^1 100\sin(n\pi x)\,dx = -\frac{200}{n\pi}\cos(n\pi x)\Big|_0^1 = \frac{200(1-(-1)^n)}{n\pi} = \begin{cases} 0 & \text{if } n=2k \text{ is even,} \\ \dfrac{400}{\pi(2k+1)} & \text{if } n=2k+1 \text{ is odd.}\end{cases}$$

Therefore $\boxed{\varphi(x) \sim \sum_{k=0}^{\infty}\frac{400}{\pi(2k+1)}\sin((2k+1)\pi x)}$ is the Fourier sine series of $\varphi(x) = 100$ on $[0,1]$.

5 (c) Since $\varphi(x) = 100$ is a continuous function on $[0,1]$ and $\varphi'(x) = 0$ is continuous (and hence piecewise continuous) on $[0,1]$, Convergence Theorem 4(i) guarantees that the Fourier sine series of φ converges to $\varphi(x) = 100$ for all x in the open interval $0 < x < 1$. Note however that the Fourier series of φ at the endpoints $x=0$ and $x=1$ do not converge to $\varphi(0) = 100$ and $\varphi(1) = 100$, respectively:

$$\sum_{k=0}^{\infty}\frac{400}{\pi(2k+1)}\sin((2k+1)\pi\, 0) = \sum_{k=0}^{\infty} 0 = 0;$$

$$\sum_{k=0}^{\infty}\frac{400}{\pi(2k+1)}\sin((2k+1)\pi) = \sum_{k=0}^{\infty} 0 = 0.$$

Therefore the Fourier sine series of φ converges to $\varphi(x)$ precisely when $\boxed{0 < x < 1}$.

5 (b) $\int_0^1 |\varphi(x)|^2 dx = \int_0^1 |100|^2 dx = 10{,}000 < \infty$ so Convergence Theorem 3 guarantees

that the Fourier sine series of φ converges to φ in the L^2-sense on $[0,1]$:

$$\lim_{N\to\infty} \int_0^1 \left|\varphi(x) - \sum_{k=0}^{N} \frac{400}{\pi(2k+1)} \sin((2k+1)\pi x)\right|^2 dx = 0 .$$

5 (d) $\boxed{\text{No,}}$ the Fourier sine series of φ does not converge uniformly to φ on $[0,1]$. To see this, note that part (c) implies

$$(*) \qquad S(\varphi;x) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{400}{\pi(2k+1)} \sin((2k+1)\pi x) = \begin{cases} 100 & \text{if } 0 < x < 1, \\ 0 & \text{if } x=0 \text{ or } x=1. \end{cases}$$

If the convergence of the Fourier sine series of φ were uniform on $[0,1]$ then the sum function $S(\varphi;\cdot)$ would be continuous on $[0,1]$. But $S(\varphi;\cdot)$ is clearly discontinuous at $x=0$ and $x=1$ so the convergence cannot be uniform.

5 (e) Applying identity $(*)$ when $x = 1/2$ yields

$$\frac{400}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \sum_{k=0}^{\infty} \frac{400}{\pi(2k+1)} \sin\left((2k+1)\frac{\pi}{2}\right) = 100,$$

and it follows that $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \boxed{\frac{\pi}{4}}$.

5 (f) Let $\{X_n\}_{n=1}^{\infty}$ be a complete orthogonal set on (a,b). If $f \in L^2(a,b)$ has Fourier series $\sum_{n=1}^{\infty} A_n X_n(x)$ with respect to $\{X_n\}_{n=1}^{\infty}$ on (a,b), then Parseval's identity holds: $\displaystyle\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$. Apply this with $\{X_n(x)\}_{n=1}^{\infty} = \{\sin(n\pi x)\}_{n=1}^{\infty}$ and $\varphi(x) = 100$ on $(0,1)$ to get $\displaystyle\frac{80{,}000}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{k=0}^{\infty} \left(\frac{400}{\pi(2k+1)}\right)^2 \int_0^1 \sin^2((2k+1)\pi x)\,dx = \int_0^1 |\varphi(x)|^2 dx$ $= 10{,}000$, and it follows that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \boxed{\frac{\pi^2}{8}}.$$

3.(40 pts.) Consider a metal rod of length 1 meter, insulated along its sides but not at its ends, which is initially at temperature 100^0 C. Suddenly both ends are inserted in an ice bath of temperature 0^0 C.
(a) Write the partial differential equation and initial/boundary conditions that model the temperature $u = u(x,t)$ of the rod at position x in [0,1] and time $t \geq 0$.
(b) Find a formula for the temperature $u = u(x,t)$. (Hint: You may find the results of problem 2 useful.)
(c) Show that the solution to the problem in part (a) is unique. (Hint: Why doesn't the maximum/minimum principle apply?)

10 (4+2+2) (a)

$$\begin{cases} u_t - ku_{xx} \overset{①}{=} 0 & \text{if } 0<x<1,\ 0<t<\infty, \\ u(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t) & \text{if } 0 \leq t < \infty, \\ u(x,0) \overset{④}{=} 100 & \text{if } 0<x<1. \end{cases}$$

20 (b) We use the method of separation of variables to solve ①-②-③-④. We seek

(1) nontrivial solutions of ①-②-③ of the form $u(x,t) = X(x)T(t)$. Substituting in ① gives $X(x)T'(t) - kX''(x)T(t) = 0$ and rearranging gives $\dfrac{-X''(x)}{X(x)} = \dfrac{-T'(t)}{kT(t)} = \text{constant} = \lambda$.
Substituting in ② and ③ gives $X(0)T(t) = 0 = X(1)T(t)$ if $0 \leq t < \infty$. Since $u(x,t) = X(x)T(t)$ is not identically zero on $0<x<1, 0<t<\infty$, it follows that $X(0) = 0 = X(1)$. Hence we have the coupled system of ODEs and BCs:

(2+1+1)

$$\begin{cases} X''(x) + \lambda X(x) \overset{⑤}{=} 0, \quad X(0) \overset{⑥}{=} 0 \overset{⑦}{=} X(1), \\ T'(t) + \lambda k T(t) \overset{⑧}{=} 0. \end{cases}$$

(1)

(1) By Section 4.1, the eigenvalue problem ⑤-⑥-⑦ has eigenvalues $\lambda_n = (n\pi)^2$ and
(2) eigenfunctions $X_n(x) = \sin(n\pi x)$ $(n = 1,2,3,\ldots)$. When $\lambda = \lambda_n = (n\pi)^2$ in ⑧,
(1) $T_n(t) = e^{-kn^2\pi^2 t}$ (up to a multiplicative constant). Therefore $u_n(x,t) = X_n(x)T_n(t) = \sin(n\pi x)e^{-kn^2\pi^2 t}$ solves ①-②-③ for $n = 1,2,3,\ldots$. The superposition principle then yields a formal solution

(4)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-kn^2\pi^2 t}$$

to ①-②-③ where $b_1, b_2, b_3, \ldots$ are arbitrary constants. To satisfy ④ we need

(1)

$$100 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{for } 0<x<1.$$

(15 pts. to here)

(15 pts. to here)

Applying problem 2, we see that we should choose the b_n's to be the Fourier sine coefficients of $\varphi(x) = 100$ on $0 < x < 1$; i.e.

(1)
$$b_n = \begin{cases} 0 & \text{if } n = 2j \text{ is even,} \\ \dfrac{400}{\pi(2j+1)} & \text{if } n = 2j+1 \text{ is odd.} \end{cases}$$

(4) Therefore $\boxed{u(x,t) = \sum_{j=0}^{\infty} \dfrac{400}{\pi(2j+1)} \sin((2j+1)\pi x) e^{-k(2j+1)^2\pi^2 t}}$ solves 1-2-3-4.

10 (c) Let $u = u_1(x,t)$ denote the solution to 1-2-3-4 obtained in part (b) of this problem and let $u = u_2(x,t)$ denote another solution. Then $v(x,t) = u_1(x,t) - u_2(x,t)$ is a continuous function in $0 \le x \le 1$, $0 \le t < \infty$, which solves

$$\begin{cases} v_t - kv_{xx} \overset{9}{=} 0 & \text{if } 0<x<1,\ 0<t<\infty, \\ v(0,t) \overset{10}{=} 0 \overset{11}{=} v(1,t) & \text{if } 0 \le t < \infty, \\ v(x,0) \overset{12}{=} 0 & \text{if } 0 < x < 1. \end{cases}$$

Let $\Theta(t) = \left(\int_0^1 (v(x,t))^2 dx\right)^{1/2}$ be the energy of v at time $t \ge 0$. Then squaring and differentiating with respect to t yields

$$2\Theta(t)\Theta'(t) = \frac{d}{dt}\int_0^1 (v(x,t)^2)dx = \int_0^1 \frac{\partial}{\partial t}(v(x,t))^2 dx = \int_0^1 2v(x,t)v_t(x,t)dx.$$

Substituting from 9 and integrating by parts gives

$$\Theta(t)\Theta'(t) = k\int_0^1 v(x,t)v_{xx}(x,t)dx = kv(x,t)v_x(x,t)\Big|_{x=0}^{1} - k\int_0^1 (v_x(x,t))^2 dx.$$

Applying 10-11 we see that $kv(x,t)v_x(x,t)\Big|_{x=0}^{1} = 0$, so

$$\Theta(t)\Theta'(t) = -k\int_0^1 v_x^2(x,t)dx \le 0.$$

Since $\Theta(t) \ge 0$, it follows that $\Theta'(t) \le 0$, i.e. Θ is decreasing on $0 \le t < \infty$, so by 12, $0 \le \Theta(t) \le \Theta(0) = \left(\int_0^1 v^2(x,0)dx\right)^{1/2} = 0$. But the vanishing theorem then yields $v(x,t) = 0$ for all $0<x<1$, $0<t<\infty$. This together with 10-11-12 imply $v(x,t) = 0$ if $0 \le x \le 1$, $0 \le t < \infty$. That is, $u_1(x,t) = u_2(x,t)$ if $0 \le x \le 1$, $0 \le t < \infty$, so the solution obtained in part (b) of this problem is unique.

Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b$$

with any symmetric boundary conditions

(2) $$\alpha_1 X(a)+\beta_1 X(b)+\gamma_1 X'(a)+\delta_1 X'(b)=0$$

(3) $$\alpha_2 X(a)+\beta_2 X(b)+\gamma_2 X'(a)+\delta_2 X'(b)=0$$

and let $\Phi=\{X_1,X_2,X_3,...\}$ be the complete orthogonal set of eigenfunctions for (1)-(2)-(3). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$f(x)\approx\sum_{n=1}^{\infty}A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle} \qquad (n=1,2,3,...).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty}A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

Math 325

Exam III

Summer 2013

number of exams: 8

mean: 63.4

median: 68

standard deviation: 15.5

Distribution of Scores:

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87–110	A	A	1
73–86	B	B	2
60–72	C	B	2
50–59	C	C	0
0–49	F	D	3