1. (33 pts.) (a) Show that $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty} = \left\{ \sin \left( \frac{(2n-1)\pi x}{2} \right) \right\}_{n=1}^{\infty}$ is an orthogonal set of functions on the interval $[0,1]$. (You may find the identity $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ useful.)

(b) Compute the Fourier coefficients $B_n(f) = \langle f, \varphi_n \rangle$ of $f(x) = x(2-x)$ with respect to the orthogonal set $\Phi$ on $[0,1]$.

(c) Show that the Fourier series of $f(x) = x(2-x)$ with respect to $\Phi$ on $[0,1]$ is given by

$$\frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}.$$  

(d) Assuming that the Fourier series of $f$ at $x$ converges to $f(x)$ for all $x$ in $[0,1]$, find the sum of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}.$$  

8 pts. (a) Let $m$ and $n$ be distinct positive integers. Then

$$\langle \varphi_n, \varphi_m \rangle = \int_0^1 \varphi_n(x) \varphi_m(x) \, dx = \int_0^1 \sin \left( \frac{(2n-1)\pi x}{2} \right) \sin \left( \frac{(2m-1)\pi x}{2} \right) \, dx = \frac{1}{2} \int_0^1 \cos \left( \frac{(n-m)\pi x}{2} \right) - \cos \left( \frac{(n+m)\pi x}{2} \right) \, dx =$$

$$= \frac{1}{2} \left[ \frac{\sin((n-m)\pi x)}{(n-m)\pi} - \frac{\sin((n+m)\pi x)}{(n+m)\pi} \right]_{x=0}^1 = 0.$$

Also, note that when $m=n$, the above calculation gives $\langle \varphi_n, \varphi_n \rangle = \frac{1}{2} \int_0^1 [1 - \cos((2n-1)\pi x)] \, dx = \frac{1}{2} \left[ x - \frac{\sin((2n-1)\pi x)}{(2n-1)\pi} \right]_{x=0}^1 = 1/2.$

(b) $B_n(f) = \langle f, \varphi_n \rangle = \int_0^1 f(x) \varphi_n(x) \, dx = \int_0^{1/2} f(x) \sin \left( \frac{(2n-1)\pi x}{2} \right) \, dx = \frac{1}{2} \int_0^{1/2} x(2-x) \sin \left( \frac{(2n-1)\pi x}{2} \right) \, dx = \int_0^{1/2} x(2-x) \sin \left( \frac{(2n-1)\pi x}{2} \right) \, dx\quad \text{Parts}$$

$$- 2 \int_0^{1/2} \left( \frac{2-2x}{2n-1} \right) \cos \left( \frac{(2n-1)\pi x}{2} \right) \, dx = - 2 \left[ - \frac{1}{2} \left(2-2x\right) \cos \left( \frac{(2n-1)\pi x}{2} \right) \right]_{x=0}^{1/2} + \int_0^{1/2} \left( \frac{2-2x}{2n-1} \right) \cos \left( \frac{(2n-1)\pi x}{2} \right) \, dx =$$

$$= \frac{32}{(2n-1)\pi^3} \quad (n = 1, 3, \ldots)$$
(c) The Fourier series of \( f(x) = x(2-x) \) with respect to \( \bar{\pi} \) on \([0,1]\) is
\[
\sum_{n=1}^{\infty} B_n(f) \varphi_n(x) = \sum_{n=1}^{\infty} \frac{32}{\pi^3(2n-1)^3} \sin\left(\frac{(2n-1)\pi x}{2}\right)
\]
\[
= \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\pi x/2\right)}{(2n-1)^3}
\]

(d) Assume
\[
f(x) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\pi x/2\right)}{(2n-1)^3}
\]
for all \( x \) in \([0,1]\). Taking \( x = 1 \), we have
\[
1 = 1(2-1) = f(1) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\pi /2\right)}{(2n-1)^3},
\]
or equivalently,
\[
\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}.
\]

\[
\begin{array}{c|c}
 n & \sin\left(\frac{(2n-1)\pi}{2}\right) \\
\hline
 1 & \sin\frac{\pi}{2} = 1 \\
 2 & \sin\frac{3\pi}{2} = -1 \\
 3 & \sin\frac{5\pi}{2} = 1 \\
 \vdots & \vdots \\
 n & (-1)^{n-1}
\end{array}
\]

\[
\sum_{n=1}^{\infty} \frac{32}{\pi^3(2n-1)^3} \sin\left(\frac{(2n-1)\pi x}{2}\right)
\]
2. (33 pts.) Solve \( u_{xx} - u_{tt} = 0 \) in the strip \( 0 < x < 1, \ 0 < t < \infty \), subject to the boundary conditions \( u(0,t) = u_x(1,t) = 0 \) if \( t \geq 0 \) and the initial conditions \( u(x,0) = x(2-x) \) and \( u_t(x,0) = 0 \) if \( 0 \leq x \leq 1 \).

(You may find the results of problem 1 useful.)

We use separation of variables. We seek nontrivial solutions of the form \( u(x,t) = X(x)T(t) \) to the homogeneous portion of the problem: 1 - 2 - 3 - 4. Applying 1 yields

\[
X(x)T''(t) - X''(x)T(t) = 0 \quad \text{for all } 0 < x < 1, \ 0 < t < \infty. \quad \text{Rearranging this leads to}
\]

\[
-\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \text{constant} = \lambda. \quad \text{Applying 2, 3, and 4 gives}
\]

\[
X(0)T(t) = 0 = X(1)T(t) \quad \text{for all } t \geq 0 \quad \text{and} \quad X(x)T'(t) = 0 \quad \text{for all } 0 \leq x \leq 1.
\]

Since \( u(x,t) = X(x)T(t) \) is not identically zero on \( 0 \leq x \leq 1, \ 0 \leq t < \infty \), we must have \( X(0) = X(1) = 0 = T'(t) \). Therefore we are led to solve the coupled system

\[
\begin{align*}
X''(x) + \lambda X(x) &= 0, \quad X(0) = X(1) = 0 = X'(1) \quad \text{(6)} \\
T''(t) + \lambda T(t) &= 0, \quad T'(t) = 0 = T'(0) \quad \text{(9)}
\end{align*}
\]

Note that (6-7-9) is an eigenvalue problem. We assume that the eigenvalues are real.

Case \( \lambda > 0 \) (say \( \lambda = \alpha^2 \) where \( \alpha > 0 \)): Then (6) becomes \( X''(x) + \alpha^2 X(x) = 0 \) with general solution \( X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \) and derivative \( X'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x) \).

Applying (7) yields \( 0 = X(0) = c_2 + c_2 \cdot 0 = c_2 \). Applying (8) gives \( 0 = X(1) = -c_1 \alpha \sin(\alpha) + c_2 \alpha \cos(\alpha) \) so \( \cos(\alpha) = 0 \) is necessary to obtain a nontrivial solution. Then

\[
\alpha = \alpha_n = \left( \frac{2n-1}{2} \right) \pi \quad \text{where } n = 1, 3, 5, \ldots \quad \text{Hence we have:}
\]

\[
\begin{align*}
\text{eigenvalues:} \quad &\lambda_n = \alpha_n^2 = \left( \frac{(2n-1)\pi}{2} \right)^2, \\
\text{eigenfunctions:} \quad &X_n(x) = \sin(\alpha_n x) = \sin\left( \frac{(2n-1)\pi x}{2} \right).
\end{align*}
\]

Case \( \lambda = 0 \): Then (6) becomes \( X''(x) = 0 \) with general solution \( X(x) = c_1 x + c_2 \) and derivative \( X'(x) = c_1 \). Applying (7) and (8) yields

\[
0 = X(0) = c_2 \quad \text{and} \quad 0 = X(1) = c_1, \quad \text{so all solutions are trivial in this case.}
\]
Case $\lambda < 0$ (say $\lambda = -\alpha^2$ where $\alpha > 0$): Then (6) becomes $\Sigma''(x) - \alpha^2 \Sigma(x) = 0$ with general solution $\Sigma(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$ and derivative $\Sigma'(x) = \alpha c_1 \sinh(\alpha x) + \alpha c_2 \cosh(\alpha x)$. Applying (7) gives $0 = \Sigma(0) = c_1 + c_2 \cdot 0 = c_1$, and (8) gives $0 = \Sigma'(0) = \alpha c_1 \sinh(\alpha x) + \alpha c_2 \cosh(\alpha x)$ so $c_2 = 0$ (since $\alpha \cosh x > 0$ when $x > 0$). Therefore all solutions are trivial in this case.

When $\lambda = \lambda_n = \alpha_n^2 = \left(\frac{(2n-1)\pi}{2}\right)^2$, (9) - (10) becomes $T_n''(t) + \alpha_n^2 T_n(t) = 0$, $T_n(0) = 0$. The general solution of (9') is $T_n(t) = c_1 \cos(\alpha_n t) + c_2 \sin(\alpha_n t)$ with derivative $T_n'(t) = -\alpha_n c_1 \sin(\alpha_n t) + \alpha_n c_2 \cos(\alpha_n t)$. Applying (10') gives $0 = T_n'(0) = -\alpha_n c_1 \cdot 0 + \alpha_n c_2 \cdot 1 = \alpha_n c_2$, so $c_2 = 0$ and $T_n(t) = \cos(\alpha_n t)$, up to a constant factor. Therefore

$$u_n(x,t) = \Sigma_n(x)T_n(t) = \sin(\alpha_n x)\cos(\alpha_n t) = \sin\left(\frac{(2n-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi t}{2}\right)$$

solves (1) - (2) - (3) - (4) for each $n=1,2,3,...$. By the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi x}{2}\right)\cos\left(\frac{(2n-1)\pi t}{2}\right)$$

is a "formal" solution of (1) - (2) - (3) - (4) for "any" choice of constants $c_1, c_2, c_3,...$. In order to satisfy (5) we must have

$$f(x) = x(2-x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

for all $0 \leq x \leq 1$. Therefore, by problem 1, we must choose $c_n = B_n(f) = \frac{32}{(2n-1)^3 \pi^3}$ for $n=1,2,3,...$. Consequently,

$$u(x,t) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x/2 \cos(2n-1)\pi t/2}{(2n-1)^3}$$

solves (1) - (2) - (3) - (4) - (5).
3. (33 pts.) (a) Show that the solution to problem 2 on this exam is unique.
(b) Is the solution to problem 2 on this exam periodic in time? That is, does there exist a number $T > 0$ such that $u(x, t+T) = u(x, t)$ for all $0 \leq x \leq 1$ and all $t \geq 0$? If not, explain why not. If so, calculate the smallest positive period of the solution.
(c) Is the solution to problem 2 on this exam bounded? If not, explain why not. If so, find the maximum and minimum values of the solution.

(a) Yes, the solution to problem 2 on this exam is unique. To see why this is true, we use the energy method. Suppose $w=V(x,t)$ is another solution to problem 2 and define

$$W(x,t) = u(x,t) - V(x,t) \quad \text{for } 0 \leq x \leq 1, \quad 0 \leq t < \infty,$$

where $u=u(x,t)$ is the solution we obtained for problem 2. Then $W$ satisfies

$$W_{tt} - W_{xx} = 0 \quad \text{if } 0 < x < 1, 0 < t < \infty,$$

$$W(0,t) = \frac{5}{2} W_x(1,t) \quad \text{if } t \geq 0,$$

$$W(x,0) = \frac{5}{2} W_t(x,0) \quad \text{if } 0 \leq x \leq 1.$$

Let $E(t) = \int_0^1 \left[ \frac{1}{2} W_t^2(x,t) + \frac{1}{2} W_x^2(x,t) \right] dx$ denote the energy function for $W$ on $0 \leq t < \infty$.

Then

$$\frac{dE}{dt} = \int_0^1 \left[ \frac{1}{2} W_t^2(x,t) + \frac{1}{2} W_x^2(x,t) \right] dx = \int_0^1 \left[ \frac{1}{2} W_t^2(x,t) + \frac{1}{2} W_x^2(x,t) \right] dx + \int_0^1 \frac{d}{dx} \left[ W_t(x,t) W_x(x,t) \right] dx - \int_0^1 W_t(x,t) W_{xx}(x,t) dx.$$

But (2) implies $w_x(0,t) = \lim_{h \to 0} \frac{W(x,t) - W(x,t)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$, so

$$W(x,t) W_t(x,t) \bigg|_{x=0} = w_x(0,t) w_t(0,t) = 0 \quad \text{by (2)}.$$

Therefore $\frac{dE}{dt} = \int_0^1 W_t(x,t) \left[ W_{tt}(x,t) - W_{xx}(x,t) \right] dx = 0$ and thus, for $t \geq 0$,

$$E(t) = E(0) = \int_0^1 \left[ \frac{1}{2} W_t^2(x,0) + \frac{1}{2} W_x^2(x,0) \right] dx = \int_0^1 \left[ \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 \right] dx = 0$$

by (4) and (5).

Therefore, the vanishing theorem implies $\frac{1}{2} W_t^2(x,t) + \frac{1}{2} W_x^2(x,t) = 0$ for $0 \leq x \leq 1$ and each fixed $t \geq 0$. Thus $W_t(x,t) = 0 = w_x(x,t)$ for all $0 \leq x \leq 1$ and all $0 \leq t < \infty$, and it follows that $W(x,t)$ is constant on $0 \leq x \leq 1, 0 \leq t < \infty$. But (5) implies this.
constant is equal to zero. That is, \(0 = u(x,t) - v(x,t)\) if \(0 \leq x \leq 1, 0 \leq t < \infty\).

This shows that \(v(x,t) = u(x,t)\) on \(0 \leq x \leq 1, 0 \leq t < \infty\), proving uniqueness of the solution \(u = u(x,t)\) to problem 2.

(b) Yes, the solution to problem 2 is periodic with smallest positive period \(T = 4\). To see this, note that

\[
\begin{align*}
\frac{u(x,t+t)}{\pi^3} = & 32 \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t + \theta/2)}{(2n-1)^3} \\
= & 32 \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t + \theta/2)}{(2n-1)^3}
\end{align*}
\]

for all \(0 \leq x \leq 1, 0 \leq t < \infty\); i.e., \(T = 4\) is a period of the solution to problem 2 in time.

Let \(T_1\) be the smallest positive period in time for the solution \(u = u(x,t)\) to problem 2. Then all positive periods in time for \(u\) are integer multiples of \(T_1\) so \(mT_1 = 4\) for some integer \(m \geq 1\). Then for every integer \(n \geq 1\) and \(t \geq 0\),

\[
\cos\left(\frac{(2n-1)\pi t}{2}\right) = \cos\left(\frac{(2n-1)\pi (t+\frac{1}{m})}{2}\right) = \cos\left(\frac{(2n-1)\pi t}{2}\right)\cos\left(\frac{(2n-1)2\pi t}{m}\right) - \sin\left(\frac{(2n-1)\pi t}{2}\right)\sin\left(\frac{(2n-1)2\pi t}{m}\right)
\]

so \(\frac{(2n-1)2\pi t}{m} = 2k\pi\) for some positive integer \(k\). In particular, when \(n = 1\),
we have \(1 \geq \frac{1}{m} = k_1 \in \mathbb{N}\) and \(m = 1\) follows; i.e., \(T_1 = 4\).

(c) Yes, the solution \(u = u(x,t)\) to problem 2 is bounded on \(0 \leq x \leq 1, 0 \leq t < \infty\), with \(u_{\max} = 1\) and \(u_{\min} = -1\). To see this, note that the function \(f\) in problems 1 and 2 can be extended to the entire real line by \(\phi(x) = \begin{cases} x(2-x) & \text{if } 0 \leq x < 2, \\ -x(2-x) & \text{if } -2 \leq x < 0, \end{cases}\) and \(\phi(x+1) = \phi(x)\) for all real \(x\).

![Graph of \(\phi(x)\)](attachment:graph.png)

Furthermore, \(-1 \leq \phi(x) \leq 1\) and \(\phi(x) = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)}{(2n-1)^3}\) for all \(-\infty < x < \infty\).

Applying the identity \(\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B) + \sin(A+B))\), we have

\[
\begin{align*}
u(x,t) &= \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\cos((2n-1)\pi t/2)}{(2n-1)^3} \\
&= \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\pi(x-t)}{(2n-1)^3} + \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x/2)\pi(x+t)}{(2n-1)^3}
\end{align*}
\]

so \(u(x,t) = \frac{1}{2} \phi(x-t) + \frac{1}{2} \phi(x+t)\) for all \(0 \leq x \leq 1\) and \(0 \leq t < \infty\). Consequently,
\[ u(x,t) \]

\[-1 = \frac{1}{2} \phi_{\min} + \frac{1}{2} \phi_{\min} \leq \frac{1}{2} \phi(x-t) + \frac{1}{2} \phi(x+t) \leq \frac{1}{2} \phi_{\max} + \frac{1}{2} \phi_{\max} = 1 \]

for all \( 0 \leq x \leq 1 \) and \( 0 \leq t < \infty \). But \( u(1,0) = f(1) = \phi(1) = 1 \) and

\[ u(1,2) = \frac{1}{2} \phi(1-2) + \frac{1}{2} \phi(1+2) = \frac{1}{2} \phi(-1) + \frac{1}{2} \phi(3-t) = \phi(-1) = -\phi(1) = -1. \]

Consequently, \( u_{\max} = 1 \) and \( u_{\min} = -1 \).
Math 325
Exam III
Summer 2014

\[ \mu = 73.0 \]
\[ \sigma = 14.3 \]
\[ n = 14 \]

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