Mathematics 325 Final Exam Name: Dr. Grow

Summer 2013

On this exam you may find one or more of the following identities useful.

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2\sin^2(\varphi)}\frac{\partial^2 u}{\partial \theta^2}$$

1.(25 pts.) Solve the partial differential equation $(1+x^2)u_x + 2xy^2u_y = 0$ subject to the auxiliary condition $u(0,y) = y^3$ if $y > 0$.

2.(25 pts.) (a) Classify the type – elliptic, hyperbolic, or parabolic – of the partial differential equation

$$u_{xx} - 2u_{xy} + u_{yy} + u_x - u_y = 4e^{y-x}.$$

(b) If possible, derive the general solution in the plane of the partial differential equation in part (a). If this is not possible, explain why.

3.(25 pts.) A homogeneous solid material occupies the region between two concentric spheres,

$$D = \{(x,y,z) \in \mathbb{R}^3 : 4 \le x^2 + y^2 + z^2 \le 100\},$$

is completely insulated, and has initial temperature at any position (x,y,z) in D given by

$$\frac{200}{\sqrt{x^2+y^2+z^2}}.$$

(a) Write the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t \ge 0$.

(b) Use the divergence theorem to help show that the total heat energy

$$H(t) = \iiint_D c\rho u(x,y,z,t)\,dV$$

of the material in D at time t is a constant function of time. Here c and ρ denote the (constant) specific heat and mass density, respectively, of the material in D.

(c) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

4.(25 pts.) Solve the heat equation in the upper half-plane

$$u_t - u_{xx} = 0 \text{ if } -\infty < x < \infty,\ 0 < t < \infty,$$

subject to the initial condition

$$u(x,0) = x^3 \text{ if } -\infty < x < \infty.$$

5.(25 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem:

$$u_t - u_{xx} + 2tu = f(x,t) \text{ if } -\infty < x < \infty,\ 0 < t < \infty,$$

$$u(x,0) = 0 \text{ if } -\infty < x < \infty.$$

6.(25 pts.) Let $\varphi(x) = 2x^2 - x^4$ for $0 \le x \le 1$.

(a) Show that the Fourier cosine series for φ on $[0,1]$ is

$$\frac{7}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^4}.$$

(b) Show that the Fourier cosine series for φ is uniformly convergent to φ on $[0,1]$.

7.(25 pts.) (a) Find a solution to the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0 \text{ if } 0 < x < 1,\ 0 < t < \infty,$$

subject to the homogeneous boundary/initial conditions

$$u_x(0,t) = 0 = u_x(1,t) \text{ if } t \ge 0 \text{ and } u_t(x,0) = 0 \text{ if } 0 \le x \le 1,$$

and the nonhomogeneous initial condition

$$u(x,0) = 2x^2 - x^4 \text{ if } 0 \le x \le 1.$$

Hint: You may find the results of problem 6 useful.

(b) Is your solution to the problem in part (a) the only one possible? Give a complete justification of your answer.

8.(25 pts.) The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Celsius on its inner boundary. On its outer boundary, the temperature distribution of the material satisfies a normal derivative condition

$$\nabla u \cdot \mathbf{n} = -\kappa$$

where κ is a positive constant.

(a) Find the steady-state temperature distribution function for the material.

(b) What are the hottest and coldest temperatures in the material? Justify your answers.

(c) Is it possible to choose κ so that the temperature on the outer boundary is 20 degrees Celsius? Support your answer with reasons.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases}1 & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases}1 & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\|\xi\|}}{a}$
D.	$\begin{cases}x & \text{if } 0<x\le b,\\ 2b-x & \text{if } b<x<2b,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases}e^{-ax} & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases}e^{ax} & \text{if } b<x<c,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases}e^{iax} & \text{if } -b<x<b,\\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases}e^{iax} & \text{if } c<x<d,\\ 0 & \text{otherwise.}\end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases}0 & \text{if } \|\xi\|\ge a,\\ \sqrt{\frac{\pi}{2}} & \text{if } \|\xi\|<a.\end{cases}$

Fourier Series Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b$$

with any symmetric boundary conditions of the form

(2) $$\begin{cases} \alpha_1 f(a)+\beta_1 f(b)+\gamma_1 f'(a)+\delta_1 f'(b)=0 \\ \alpha_2 f(a)+\beta_2 f(b)+\gamma_2 f'(a)+\delta_2 f'(b)=0 \end{cases}$$

and let $\Phi=\{X_1,X_2,X_3,\ldots\}$ be the complete orthogonal set of eigenfunctions for (1)-(2). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle} \qquad (n=1,2,3,\ldots).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

#1 $\quad (1+x^2)u_x + 2xy^2u_y = 0, \qquad u(0,y) = y^3 \text{ if } y>0.$

The characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ are given by $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$. In our case, this is $\dfrac{dy}{dx} = \dfrac{2xy^2}{1+x^2} \Rightarrow \displaystyle\int \frac{1}{y^2}dy = \int \frac{2x\,dx}{1+x^2}$

so $-\dfrac{1}{y} = \ln(1+x^2) + c \iff y = \dfrac{-1}{c+\ln(1+x^2)}$. Along any characteristic curve, the solution $u=u(x,y)$ of the p.d.e. is constant. Along any such curve

$$u(x,y(x)) = u(0,y(0)) = u\left(0,-\frac{1}{c}\right) = f(c).$$

The general solution of the p.d.e. is thus

$$u(x,y) = f\left(-\frac{1}{y} - \ln(1+x^2)\right)$$

where f is an arbitrary continuously differentiable function of a single real variable. We apply the auxiliary condition to obtain

$$y^3 = u(0,y) = f\left(-\frac{1}{y} - \ln(1+0)\right) = f\left(\frac{-1}{y}\right) \qquad \text{if } y>0.$$

Setting $\star = -\dfrac{1}{y}$ we have $f(\star) = \left(-\dfrac{1}{\star}\right)^3 = -\dfrac{1}{\star^3}$ for $\star < 0$.

Therefore

$$u(x,y) = f\left(\frac{-1}{y} - \ln(1+x^2)\right) = \frac{-1}{\left(-\frac{1}{y} - \ln(1+x^2)\right)^3}$$

$$\boxed{u(x,y) = \frac{1}{\left[\frac{1}{y} + \ln(1+x^2)\right]^3}} \qquad \text{if } y>0,$$

or equivalently,

$$\boxed{u(x,y) = \frac{y^3}{\left[1 + y\ln(1+x^2)\right]^3}} \qquad \text{if } y>0.$$

#2 $\quad u_{xx} - 2u_{xy} + u_{yy} + u_x - u_y = 4e^{y-x}$

(2) (a) $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$ The p.d.e. is $\boxed{\text{parabolic}}$.

(b) We "factor" the operators in the p.d.e. as follows.

$$\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right)u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 4e^{y-x}$$

(4) $$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = 4e^{y-x}.$$

(8) Let $\begin{cases} \xi = -(\beta x - \alpha y) = -(-x-y) = x+y, \\ \eta = \alpha x + \beta y = x - y. \end{cases}$

The chain rule implies these differential operator expressions:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}.$$

Therefore $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right) - \left(\cancel{\frac{\partial}{\partial \xi}} - \frac{\partial}{\partial \eta}\right) = 2\frac{\partial}{\partial \eta}$ so the p.d.e. can be rewritten in the $\xi\eta$- coordinates as

(12) $(*)$ $$4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} = 4e^{-\eta}$$

[For an alternate solution see the bottom of the next page.]

Letting $\frac{\partial u}{\partial \eta} = v$, this reduces the p.d.e. to a first-order problem:

$$4\frac{\partial v}{\partial \eta} + 2v = 4e^{-\eta}$$

(13) $$\frac{\partial v}{\partial \eta} + \frac{1}{2}v = e^{-\eta}.$$

(15) An integrating factor is $\mu(\eta) = e^{\int \frac{1}{2} d\eta} = e^{\frac{1}{2}\eta}$ so multiplying through

by the integrating factor yields

(16) $$e^{\frac{1}{2}\eta}\frac{\partial v}{\partial \eta} + \frac{1}{2}e^{\frac{1}{2}\eta}v = e^{-\frac{\eta}{2}}.$$

But the left member is an exact expression: $\frac{\partial}{\partial \eta}\left(e^{\frac{1}{2}\eta}v\right) = e^{\frac{1}{2}\eta}\frac{\partial v}{\partial \eta} + \frac{1}{2}e^{\frac{1}{2}\eta}v$.

Therefore we have

$$\frac{\partial}{\partial \eta}\left(e^{\frac{1}{2}\eta}v\right) = e^{-\frac{1}{2}\eta}$$

and integration yields

(18) $$e^{\frac{1}{2}\eta}v = \int e^{-\frac{1}{2}\eta}d\eta = -2e^{-\frac{1}{2}\eta} + c_1(\xi)$$

so

(20) $$v = -2e^{-\eta} + c_1(\xi)e^{-\frac{1}{2}\eta}.$$

But $v = \frac{\partial u}{\partial \eta}$, so substituting in the equation above and integrating gives

$$u = \int \frac{\partial u}{\partial \eta}d\eta = \int\left(-2e^{-\eta} + c_1(\xi)e^{-\frac{1}{2}\eta}\right)d\eta$$

(24) $$u = 2e^{-\eta} - 2c_1(\xi)e^{-\frac{1}{2}\eta} + c_2(\xi)$$

As a function of x and y, we have

(25) $$\boxed{u(x,y) = 2e^{y-x} + f(x+y)e^{\frac{1}{2}(y-x)} + g(x+y)}$$

where f and g are arbitrary twice continuously differentiable functions of a single real variable.

Alternate solution method starting at (*) $4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} = 4e^{-\eta}$: (12 pts. to here)

The general solution of the second order linear equation (*) has the form

(13) $u(\eta) = u_c(\eta) + u_p(\eta)$ where u_c is the general solution of the associated

homogeneous equation $4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} \overset{(**)}{=} 0$ and u_p is any particular solution (i.e. without arbitrary constants) of the nonhomogeneous equation (*). We look for exponential function solutions to (**): $u = e^{r\eta}$ where r is a constant. Then $\frac{\partial u}{\partial \eta} = re^{r\eta}$ and $\frac{\partial^2 u}{\partial \eta^2} = r^2 e^{r\eta}$, so substituting in (**) gives

$$4r^2e^{r\eta} + 2re^{r\eta} = 0$$

$$\Rightarrow \quad 2e^{r\eta}(2r+1)r = 0.$$

Therefore $r = 0$ or $r = -\frac{1}{2}$. Hence $u_c = c_1(\xi)e^{-\eta/2} + c_2(\xi)$ where c_1 and c_2 are arbitrary "constants" which may vary with the parameter ξ.

To find a particular solution of (*), we use the method of undetermined coefficients. Looking at the right member of (*), we expect a particular solution of the form $u_p = Ae^{-\eta}$ where A is a constant to be determined. Then $\frac{\partial u_p}{\partial \eta} = -Ae^{-\eta}$ and $\frac{\partial^2 u_p}{\partial \eta^2} = Ae^{-\eta}$. We want to choose A so that u_p solves (*):

$$4\frac{\partial^2 u_p}{\partial \eta^2} + 2\frac{\partial u_p}{\partial \eta} = 4e^{-\eta}$$

$$4\left(Ae^{-\eta}\right) + 2\left(-Ae^{-\eta}\right) = 4e^{-\eta}$$

$$2Ae^{-\eta} = 4e^{-\eta}$$

so we get a solution by choosing $A = 2$. Therefore

$$u = u_c + u_p = c_1(\xi)e^{-\eta/2} + c_2(\xi) + 2e^{-\eta}.$$

[The solution is finished as before.]

#3

A p.d.e. and initial/boundary conditions governing the temperature $u = u(x,y,z,t)$ are:

(9) (a)

$$
\begin{cases}
u_t - k\big(\overbrace{u_{xx}+u_{yy}+u_{zz}}^{\nabla^2 u}\big) \overset{(1)}{=} 0 & \text{if } 4 < x^2+y^2+z^2 < 100 \text{ and } 0<t<\infty, \\
\nabla u \cdot \vec{n} \overset{(2)}{=} 0 & \text{if } x^2+y^2+z^2 = 4 \text{ or } 100 \text{ and } t \geq 0 \\
u(x,y,z,0) \overset{(3)}{=} \dfrac{200}{\sqrt{x^2+y^2+z^2}} & \text{if } 4 \leq x^2+y^2+z^2 \leq 100 .
\end{cases}
$$

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(7) (b)

$$\frac{dH}{dt} = \frac{d}{dt}\iiint_D c\rho u \, dV = \iiint_D c\rho u_t \, dV \overset{\text{by (1)}}{=} \iiint_D kc\rho \nabla^2 u \, dV = \iiint_D kc\rho \nabla\cdot(\nabla u)\, dV$$

$$\underset{\substack{\text{By Gauss'}\\ \text{Divergence}\\ \text{Theorem}}}{=} \iint_{\partial D} kc\rho \nabla u \cdot \vec{n}\, dS \underset{\text{by (2)}}{=} \iint_{\partial D} kc\rho(0)\, dS = 0 .$$

Therefore $\boxed{H(t) = H(0) = \text{constant for all } t>0.}$

(9) (c) Let $U_\infty = \lim_{t\to\infty} u(x,y,z,t)$ be the (constant) steady-state temperature at all points in D. Since $H(0) = H(t)$ for all $t>0$,

$$H(0) = \lim_{t\to\infty} H(t) = \lim_{t\to\infty}\iiint_D c\rho u\, dV = \iiint_D c\rho\big(\lim_{t\to\infty} u\big)\, dV = \iiint_D c\rho U_\infty \, dV$$

$$= c\rho U_\infty \operatorname{vol}(D) = c\rho U_\infty\left(\tfrac{4}{3}\pi(10)^3 - \tfrac{4}{3}\pi(2)^3\right) = c\rho U_\infty \tfrac{4}{3}\pi(992).$$

But
$$H(0) = \iiint_D c\rho u(x,y,z,0)\, dV \overset{\text{by (3)}}{=} \iiint_D c\rho \frac{200}{\sqrt{x^2+y^2+z^2}}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{10} c\rho \frac{200}{r} r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= c\rho\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\varphi\, d\varphi\right)\left(\int_2^{10} 200 r\, dr\right) = c\rho(2\pi)(2)\left(100r^2\Big|_2^{10}\right) = c\rho(400\pi)(96)$$

Thus

$$c\rho(400\pi)(96) = c\rho U_\infty \frac{4}{3}\pi(992)$$

$$\Rightarrow \quad U_\infty = \frac{(\cancel{400}^{100})(\overbrace{96}^{\cancel{2}^{8}\cdot 3})}{\underbrace{992}_{\cancel{2}^{5}\cdot 31}} \cdot \frac{3}{\cancel{4}} = \boxed{\frac{900}{31}}$$

(This is approximately 29.)

#4 $\begin{cases} u_t - u_{xx} = 0 & \text{if } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = x^3 & \text{if } -\infty < x < \infty. \end{cases}$

By the formula for solution to the heat equation in the upper half-plane,

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy,$$

with $k=1$ and $\varphi(y) = y^3$, we have

$$(*) \qquad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^3\,dy.$$

(Actually, the formula is only guaranteed to yield a solution provided φ is bounded and continuous on the entire real line. Since our function $\varphi(y)=y^3$ is not bounded, we will need to check our answer.)

Let $p = \frac{y-x}{\sqrt{4t}}$ in the integral in $(*)$. Then $dp = \frac{dy}{\sqrt{4t}}$ and $\sqrt{4t}\,p + x = y$ so

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (\sqrt{4t}\,p + x)^3\,dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left\{ (\sqrt{4t}\,p)^3 + 3(\sqrt{4t}\,p)^2 x + 3(\sqrt{4t}\,p)x^2 + x^3 \right\} dp$$

$$= \frac{4t\sqrt{4t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^3\,dp + \frac{12tx}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp + \frac{3\sqrt{4t}\,x^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p\,dp + \frac{x^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp$$

But $\int_{-\infty}^{\infty} e^{-p^2} p^3\,dp = 0 = \int_{-\infty}^{\infty} e^{-p^2} p\,dp$ since the integrands are odd functions,

$\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}$, and $\int_{-\infty}^{\infty} e^{-p^2} p^2\,dp = \frac{\sqrt{\pi}}{2}$ (given identity) so

$$\boxed{u(x,t) = x^3 + 6tx}$$

Check: $u_t - u_{xx} = 6x - 6x \overset{\checkmark}{=} 0$, $u(x,0) \overset{\checkmark}{=} x^3$.

#5
$$\begin{cases} u_t - u_{xx} + 2tu \overset{①}{=} f(x,t) & \text{if } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) \overset{②}{=} 0 & \text{if } -\infty < x < \infty. \end{cases}$$

Suppose $u = u(x,t)$ is a solution to ①-② and take the Fourier transform (with respect to x) of ①:

$$\mathcal{F}\{u_t - u_{xx} + 2tu\}(\xi) = \mathcal{F}(f(\cdot,t))(\xi).$$

Then

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + 2t\mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$

$$(*) \quad \frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+2t)\mathcal{F}(u)(\xi) = \hat{f}(\xi,t).$$

This is a first order linear differential equation in the independent variable t.
An integrating factor is

$$\mu(t) = e^{\int(\xi^2+2t)\,dt} = e^{\xi^2 t + t^2}.$$

Multiplying $(*)$ by the integrating factor and combining terms gives

$$\frac{\partial}{\partial t}\left(e^{\xi^2t+t^2}\mathcal{F}(u)(\xi)\right) = e^{\xi^2t+t^2}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+2t)e^{\xi^2t+t^2}\mathcal{F}(u)(\xi) = e^{\xi^2t+t^2}\hat{f}(\xi,t).$$

Integrating yields

$$e^{\xi^2t+t^2}\mathcal{F}(u)(\xi) = \int_0^t e^{\xi^2\tau+\tau^2}\hat{f}(\xi,\tau)\,d\tau + c(\xi).$$

Applying ② to the equation above leads to

$$0 = e^{\xi^2t+t^2}\mathcal{F}(u)(\xi)\Big|_{t=0} = \left(\int_0^t e^{\xi^2\tau+\tau^2}\hat{f}(\xi,\tau)\,d\tau + c(\xi)\right)\Big|_{t=0} = c(\xi)$$

Thus

$$(**) \quad \mathcal{F}(u)(\xi) = e^{-t^2-\xi^2t}\int_0^t e^{\xi^2\tau+\tau^2}\cdot\hat{f}(\xi,\tau)\,d\tau = \int_0^t e^{\tau^2-t^2}\cdot e^{-\xi^2(t-\tau)}\hat{f}(\xi,\tau)\,d\tau$$

Applying entry I in the Fourier transform table:

$$\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}$$

with $t-\tau = \frac{1}{4a}$, or equivalently $a = \frac{1}{4(t-\tau)}$, gives

(15) $$\mathcal{F}\left(\frac{1}{\sqrt{2(t-\tau)}} e^{-\frac{(\cdot)^2}{4(t-\tau)}} \right)(\xi) = e^{-\xi^2(t-\tau)} .$$

Substituting this in (**) produces

(16) $$\mathcal{F}(u)(\xi) = \int_0^t e^{\tau^2 - t^2} \mathcal{F}\left(\frac{1}{\sqrt{2(t-\tau)}} e^{-\frac{(\cdot)^2}{4(t-\tau)}} \right)(\xi)\, \mathcal{F}\big(f(\cdot,\tau)\big)(\xi)\, d\tau .$$

The convolution property of Fourier transforms, $\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f*g)(\xi)$, then implies

$$\mathcal{F}(u)(\xi) = \int_0^t e^{\tau^2-t^2} \mathcal{F}\left(\frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau) \right)(\xi)\, d\tau$$

(18) $$= \int_0^t \mathcal{F}\left(\frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau) \right)(\xi)\, d\tau .$$

Interchanging the order of the integrations — recall that the Fourier transform is an integral with respect to x over the interval $-\infty < x < \infty$ — yields

(20) $$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\int_0^t \frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau)\, d\tau \right)(\xi)$$

and consequently the inversion theorem gives

(22) $$u(x,t) = \int_0^t \frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} \left(e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau) \right)(x)\, d\tau$$

(25) $$= \int_0^t \frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-\tau)}} f(y,\tau)\, dy\, d\tau$$

$$= \boxed{\int_{-\infty}^{\infty} \int_0^t \frac{e^{-\frac{(x-y)^2}{4(t-\tau)}} \cdot e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} f(y,\tau)\, d\tau\, dy}$$

$\underline{\#6}$ $\quad \varphi(x) = 2x^2 - x^4$ if $0 \le x \le 1$.

(15) (a) The Fourier cosine series for a function φ on $(0, \ell)$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$$

where
$$a_n = \frac{\langle \varphi, \cos(n\pi(\cdot)/\ell)\rangle}{\langle \cos(n\pi(\cdot)/\ell), \cos(n\pi(\cdot)/\ell)\rangle} = \frac{2}{\ell}\int_0^{\ell} \varphi(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad (n \ge 1)$$

and
$$a_0 = \frac{\langle \varphi, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{\ell}\int_0^{\ell}\varphi(x)\,dx.$$

In our case $\ell = 1$ so

(2) $$a_0 = \int_0^1 \varphi(x)dx = \int_0^1 (2x^2 - x^4)dx = \left(\frac{2x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{10-3}{15} = \frac{7}{15}$$

and

(3) $$a_n = 2\int_0^1 \overbrace{\varphi(x)}^{v}\,\overbrace{\cos(n\pi x)dx}^{dv}$$

(4) $$= 2\left[\cancel{\frac{\varphi(x)\sin(n\pi x)}{n\pi}\Bigg|_0^1}^{\,0} - \frac{1}{n\pi}\int_0^1 \overbrace{\varphi'(x)}^{v}\,\overbrace{\sin(n\pi x)dx}^{dv}\right]$$

(6) $$= \frac{-2}{n\pi}\left[\cancel{-\frac{\varphi'(x)\cos(n\pi x)}{n\pi}\Bigg|_0^1}^{\,0} + \frac{1}{n\pi}\int_0^1 \underbrace{\varphi''(x)}_{v}\,\underbrace{\cos(n\pi x)dx}_{dv}\right]$$

(7) $$= \frac{-2}{(n\pi)^2}\left[\cancel{\frac{\varphi''(x)\sin(n\pi x)}{n\pi}\Bigg|_0^1}^{\,0} - \frac{1}{n\pi}\int_0^1 \underbrace{\varphi'''(x)}_{v}\,\underbrace{\sin(n\pi x)dx}_{dv}\right]$$

(9) $$= \frac{2}{(n\pi)^3}\left[-\frac{\varphi'''(x)\cos(n\pi x)}{n\pi}\Bigg|_0^1 + \cancel{\frac{1}{n\pi}\int_0^1 \varphi^{(4)}(x)\cos(n\pi x)dx}^{\,0}\right]$$

[since $\varphi^{(4)}$ is constant and $\int_0^1 \cos(n\pi x)dx = 0$

(10) $$= \frac{2}{(n\pi)^4}\left(24x\cos(n\pi x)\Big|_0^1\right)$$

(11) $$= \frac{48(-1)^n}{(n\pi)^4} \quad \text{if } n \ge 1.$$

Note that
$\varphi'(x) = 4x - 4x^3$
so $\varphi'(0) = 0 = \varphi'(1)$.
$\varphi''(x) = 4 - 12x^2$
$\varphi'''(x) = -24x$
$\varphi^{(4)}(x) = -24$

Therefore the Fourier cosine series for φ on $[0,1]$ is

(15) $$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \boxed{\frac{7}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^4}}.$$

(10) (b) $\Phi = \{1, \cos(\pi x), \cos(2\pi x), \cos(3\pi x), \ldots\}$ is the complete orthogonal set of eigenfunctions for

$$(1) \qquad X''(x) + \lambda X(x) = 0 \quad \text{in} \quad 0 < x < 1$$

with the symmetric (Neumann) boundary conditions

(2) $$(2) \qquad X'(0) = 0 \quad \text{and} \quad X'(1) = 0.$$

(3) Note : $\varphi(x) = 2x^2 - x^4$

(4) $\varphi'(x) = 4x - 4x^3$

(6) $\varphi''(x) = 4 - 12x^2$

These derivatives exist and are continuous for $0 \le x \le 1$.

(9) $\varphi'(0) = 0 = \varphi'(1)$ so φ satisfies the boundary conditions (2) as well. Thus Theorem 2 implies that the Fourier cosine series

(10) of φ converges uniformly to φ on $[0,1]$.

#7

$$\begin{cases} u_{tt} - u_{xx} \overset{(1)}{=} 0 & \text{if } 0 < x < 1,\ 0 < t < \infty, \\ u_x(0,t) \overset{(2)}{=} 0 \overset{(3)}{=} u_x(1,t) & \text{if } t \geq 0, \\ u_t(x,0) \overset{(4)}{=} 0 & \text{if } 0 \leq x \leq 1, \\ u(x,0) \overset{(5)}{=} 2x^2 - x^4 & \text{if } 0 \leq x \leq 1. \end{cases}$$

(15) (a) We use separation of variables. We seek nontrivial solutions of (1)-(2)-(3)-(4) of the form $u(x,t) = X(x)T(t)$. Substituting in (1) gives

$$X(x)T''(t) - X''(x)T(t) = 0 \quad \Rightarrow \quad -\frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} = \text{constant} = \lambda.$$

Substituting in (2)-(3) yields

$$X'(0)T(t) = 0 \quad \text{and} \quad X'(1)T(t) = 0 \quad \text{if } t \geq 0.$$

In order to have a nontrivial solution, it follows that $X'(0) = X'(1) = 0$.
Similarly, substituting in (4) gives

$$X(x)T'(0) = 0 \quad \text{if } 0 \leq x \leq 1$$

and hence $T'(0) = 0$ in order to have a nontrivial solution. Collecting these facts, we have

$$\begin{cases} X''(x) + \lambda X(x) \overset{(6)}{=} 0, \quad X'(0) \overset{(7)}{=} 0 \overset{(8)}{=} X'(1), \\ T''(t) + \lambda T(t) \overset{(9)}{=} 0, \quad T'(0) \overset{(10)}{=} 0. \end{cases}$$

(Note that (6)-(7)-(8) is an eigenvalue problem.) In the textbook, section 4.2, it is shown that the eigenvalues and eigenfunctions of $T = -\frac{d^2}{dx^2}$ with Neumann boundary conditions (7)-(8) are, respectively,

$$\lambda_n = (n\pi)^2 \quad \text{and} \quad X_n(x) = \cos(n\pi x), \quad (n = 0, 1, 2, \ldots).$$

Substituting $\lambda = \lambda_n = (n\pi)^2$ in (9), we find it's general solution is

$$T_n(t) = \alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t).$$

Applying (10), we see that $T_n(t) = \cos(n\pi t)$ for $n = 0, 1, 2, 3, \ldots$ (up to a constant multiple, anyway).

Consequently, $u_n(x,t) = X_n(x)T_n(t) = \cos(n\pi x)\cos(n\pi t) \quad (n = 0,1,2,\dots)$ solves (1)-(2)-(3)-(4). The superposition principle then gives a formal soluton

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\cos(n\pi t)$$

to (1)-(2)-(3)-(4) for any choice of constants $a_0, a_1, a_2, \dots$. Applying (5) yields

$$2x^2 - x^4 = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \qquad \text{if } 0 \le x \le 1.$$

By problem #6, we may take

$$a_0 = \frac{7}{15} \quad \text{and} \quad a_n = \frac{48(-1)^n}{\pi^4 n^4} \quad \text{if } n \ge 1.$$

Hence a solution to (1)-(2)-(3)-(4)-(5) is

$$u(x,t) = \frac{7}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)\cos(n\pi t)}{n^4}.$$

(10) (b) The solution to (1)-(2)-(3)-(4)-(5) obtained above is unique. To see this, suppose that $u = v(x,t)$ were another solution and define $w(x,t) = u(x,t) - v(x,t)$. Then w solves

$$\begin{cases} w_{tt} - w_{xx} \overset{(11)}{=} 0 & \text{if } 0 < x < 1,\ 0 < t < \infty, \\ w_x(0,t) \overset{(12)}{=} 0 \overset{(13)}{=} w_x(1,t) & \text{if } t \ge 0, \\ w(x,0) \overset{(14)}{=} 0 \overset{(15)}{=} w_t(x,0) & \text{if } 0 \le x \le 1. \end{cases}$$

Let

$$E(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]\,dx \qquad (t \ge 0)$$

be the energy function of w on $[0,\infty)$. Then

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]\,dx\right)$$

(4) $$\frac{dE}{dt} = \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\left[w_t^2(x,t) + w_x^2(x,t)\right]dx = \int_0^1 \left[w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_{xt}(x,t)\right]dx .$$

Using ⑪ to substitute for w_{tt} in the integrand of the left member of the equation above and integrating by parts the second term in the integrand with $U = w_x(x,t)$ and $dV = w_{xt}(x,t)\,dx$, we find

(6) $$\frac{dE}{dt} = \int_0^1 w_t(x,t)w_{xx}(x,t)\,dx + w_x(x,t)w_t(x,t)\Big|_{x=0}^{1} - \int_0^1 w_t(x,t)w_{xx}(x,t)\,dx$$

$$= w_x(1,t)w_t(1,t) - w_x(0,t)w_t(0,t)$$

(7) $$= 0$$

by ⑫ and ⑬. Therefore the energy of w is a constant function on $[0,\infty)$ so

(8) $$0 \le E(t) = E(0) = \int_0^1 \left[w_t^2(x,0) + w_x^2(x,0)\right]dx = 0$$

by ⑮ and differentiation of ⑭: $w(x,0) = 0$ if $0 \le x \le 1$ implies $w_x(x,0) = 0$ if $0 \le x \le 1$. Consequently, the vanishing theorem yields that $w_t^2(x,t) = 0 = w_x^2(x,t)$ for all $0 \le x \le 1$ and each $t \ge 0$ so

(9) $w(x,t) =$ constant in $0 \le x \le 1, 0 \le t < \infty$. But ⑭ then gives $w(x,t) = 0$

(10) if $0 \le x \le 1, 0 \le t < \infty$. That is, $u(x,t) = v(x,t)$ for all $0 \le x \le 1$ and $0 \le t < \infty$.

#8 (a) The steady-state temperature distribution $u=u(x,y,z)$ of the material in the spherical shell satisfies

$$\begin{cases} \nabla^2 u \overset{①}{=} 0 & \text{if } 1 < x^2+y^2+z^2 < 4, \\ u \overset{②}{=} 100 & \text{if } x^2+y^2+z^2 = 1, \\ \nabla u \cdot \vec{n} \overset{③}{=} -K & \text{if } x^2+y^2+z^2 = 4. \end{cases}$$

Since the region, partial differential equation, and boundary conditions are invariant under rotations about the origin, we expect that any solution to ①-②-③ at each point depends only on the distance of that point from the origin. I.e., in spherical coordinates we expect $u=u(r)$, independent of θ and φ. Then $\partial u/\partial\varphi = 0$ and $\partial^2 u/\partial\theta^2 = 0$ so ① becomes

$$0 = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial\varphi}\left(\sin(\varphi)\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\cdot\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

Multiplying by r^2 and integrating yields

$$(*) \quad c_1 = r^2\frac{\partial u}{\partial r}.$$

In spherical coordinates, ③ becomes $\frac{\partial u}{\partial r} = -K$ if $r=2$. Consequently $c_1 = r^2\frac{\partial u}{\partial r}\Big|_{r=2} = -4K$. Substituting in (*) gives

$$-4K = r^2\frac{\partial u}{\partial r}.$$

Rearranging and integrating yields

$$u = \int \frac{\partial u}{\partial r}\,dr = \int \frac{-4K}{r^2}\,dr = \frac{4K}{r} + c_2.$$

Applying ② produces

$$100 = u\Big|_{r=1} = \left(\frac{4K}{r} + c_2\right)\Big|_{r=1} = 4K + c_2$$

so $c_2 = 100 - 4K$ and $\boxed{u(r) = \frac{4K}{r} + 100 - 4K}$ if $1 \le r \le 2$.

(This solution is unique. See the note on the last page of these solutions.)

(b) Since $u(r) = \frac{4K}{r} + 100 - 4K$ is harmonic in the region $1 < r < 2$ and continuous on its closure $1 \le r \le 2$, the maximum/minimum principle states the hottest and coldest temperatures must occur on the boundaries of the region. Hence

$$u_{max} = u(1) = \frac{4K}{1} + 100 - 4K = 100$$

and

$$u_{min} = u(2) = \frac{4K}{2} + 100 - 4K = 100 - 2K.$$

(c) If we choose $K = 40$, then the work in part (b) shows that $u(2) = 100 - 2K = 100 - 80 = 20^\circ$ Celsius. Therefore it is possible to choose K so the temperature on the outer boundary is 20 degrees Celsius.

in #8

Note: The solution to 1-2-3 in #8 is unique. To see this, suppose that $u=u(x,y,z)$ and $u=v(x,y,z)$ were solutions to 1-2-3 and let

$$w(x,y,z) = u(x,y,z) - v(x,y,z).$$

Then w solves

$$\begin{cases} \nabla^2 w \overset{1'}{=} 0 & \text{if } 1<r<2, \\ w \overset{2'}{=} 0 & \text{if } r=1, \\ \vec{\nabla} w \cdot \vec{n} \overset{3'}{=} 0 & \text{if } r=2. \end{cases}$$

Let $E = \iiint\limits_D |\vec{\nabla} w|^2 dV$ where $D = \{(x,y,z) \in \mathbb{R}^3 : 1 \le x^2+y^2+z^2 \le 4\}$.

Recall the identity $\vec{\nabla}\cdot(f\vec{\nabla}g) = \vec{\nabla}f\cdot\vec{\nabla}g + f\nabla^2 g$. Rearranging leads to $\vec{\nabla}f\cdot\vec{\nabla}g = \vec{\nabla}\cdot(f\vec{\nabla}g) - f\nabla^2 g$ and setting $f=g=w$ gives

$$|\vec{\nabla}w|^2 = \vec{\nabla}w\cdot\vec{\nabla}w = \vec{\nabla}\cdot(w\vec{\nabla}w) - w\nabla^2 w.$$

Substituting in the expression for E and using 1′ produces

$$E = \iiint\limits_D \left[\vec{\nabla}\cdot(w\vec{\nabla}w) - w\nabla^2 w\right] dV = \iiint\limits_D \vec{\nabla}\cdot(w\vec{\nabla}w)\,dV.$$

Applying Gauss' divergence theorem yields

$$E = \iint\limits_{\partial D} w\vec{\nabla}w\cdot\vec{n}\,dS.$$

On the $r=1$ portion of ∂D, 2′ implies $w\vec{\nabla}w\cdot\vec{n}=0$. On the $r=2$ portion of ∂D, 3′ implies $w\vec{\nabla}w\cdot\vec{n}=0$. Hence $E=0$ and the vanishing theorem implies $|\vec{\nabla}w|=0$ in D. Consequently $w=$ constant in D. Condition 2′ then yields $w=0$ in D; i.e. $u(x,y,z)=v(x,y,z)$ for all (x,y,z) in D.

Math 325

Final Exam

Summer 2013

number taking exam = 8

median = 139

mean = 134.9

standard deviation = 28.8

Distribution of Scores

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
174-200	A	A	1
146-173	B	B	2
120-145	C	B	3
100-119	C	C	0
0-99	F	D	2