On this exam you may find one or more of the following identities useful.

\[ \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp = \frac{\sqrt{\pi}}{2} \]

\[ \nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin(\varphi)} \frac{\partial}{\partial \varphi} \left( \sin(\varphi) \frac{\partial u}{\partial \varphi} \right) + \frac{1}{r^2 \sin^2(\varphi)} \frac{\partial^2 u}{\partial \theta^2} \]

1. (25 pts.) Solve the partial differential equation \((1 + x^2)u_x + 2xy^2u_y = 0\) subject to the auxiliary condition \(u(0, y) = y^3\) if \(y > 0\).

2. (25 pts.) (a) Classify the type – elliptic, hyperbolic, or parabolic – of the partial differential equation

\[ u_{xx} - 2u_{xy} + u_{yy} + u_x - u_y = 4e^{x-y} \]

(b) If possible, derive the general solution in the plane of the partial differential equation in part (a). If this is not possible, explain why.

3. (25 pts.) A homogeneous solid material occupies the region between two concentric spheres,

\[ D = \{(x, y, z) \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 100 \} \]

is completely insulated, and has initial temperature at any position \((x, y, z)\) in \(D\) given by

\[ \frac{200}{\sqrt{x^2 + y^2 + z^2}} \]

(a) Write the partial differential equation and initial/boundary conditions that completely govern the temperature \(u(x, y, z, t)\) at position \((x, y, z)\) in \(D\) and time \(t \geq 0\).

(b) Use the divergence theorem to help show that the total heat energy

\[ H(t) = \iiint_D c \rho u(x, y, z, t) \, dV \]

of the material in \(D\) at time \(t\) is a constant function of time. Here \(c\) and \(\rho\) denote the (constant) specific heat and mass density, respectively, of the material in \(D\).

(c) Compute the (constant) steady-state temperature that the material in \(D\) reaches after a long time.

4. (25 pts.) Solve the heat equation in the upper half-plane

\[ u_t - u_{xx} = 0 \quad \text{if} \quad -\infty < x < \infty, \quad 0 < t < \infty, \]

subject to the initial condition

\[ u(x, 0) = x^3 \quad \text{if} \quad -\infty < x < \infty. \]

5. (25 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem:

\[ u_t - u_{xx} + 2u = f(x, t) \quad \text{if} \quad -\infty < x < \infty, \quad 0 < t < \infty, \]

\[ u(x, 0) = 0 \quad \text{if} \quad -\infty < x < \infty. \]
6. (25 pts.) Let \( \varphi(x) = 2x^2 - x^4 \) for \( 0 \leq x \leq 1 \).

(a) Show that the Fourier cosine series for \( \varphi \) on \([0,1]\) is

\[
\frac{7}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^4}.
\]

(b) Show that the Fourier cosine series for \( \varphi \) is uniformly convergent to \( \varphi \) on \([0,1]\).

7. (25 pts.) (a) Find a solution to the one-dimensional wave equation

\[
u_t - \nu_{xx} = 0 \quad \text{if} \quad 0 < x < 1, \quad 0 < t < \infty,
\]

subject to the homogeneous boundary/initial conditions

\[
u_x(0,t) = 0 = \nu_x(1,t) \quad \text{if} \quad t \geq 0 \quad \text{and} \quad \nu_t(x,0) = 0 \quad \text{if} \quad 0 \leq x \leq 1,
\]

and the nonhomogeneous initial condition

\[
u(x,0) = 2x^2 - x^4 \quad \text{if} \quad 0 \leq x \leq 1.
\]

Hint: You may find the results of problem 6 useful.

(b) Is your solution to the problem in part (a) the only one possible? Give a complete justification of your answer.

8. (25 pts.) The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Celsius on its inner boundary. On its outer boundary, the temperature distribution of the material satisfies a normal derivative condition

\[\nabla \nu \cdot \mathbf{n} = -\kappa\]

where \( \kappa \) is a positive constant.

(a) Find the steady-state temperature distribution function for the material.

(b) What are the hottest and coldest temperatures in the material? Justify your answers.

(c) Is it possible to choose \( \kappa \) so that the temperature on the outer boundary is 20 degrees Celsius? Support your answer with reasons.
A Brief Table of Fourier Transforms

\[ f(x) \quad \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx \]

A. \[
\begin{cases} 
1 & \text{if } -b < x < b, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
\sqrt{2} \sin(b\xi) \quad \xi \sqrt{\pi}
\]

B. \[
\begin{cases} 
1 & \text{if } c < x < d, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
e^{-i\xi} - e^{-i\xi^2} \quad i\xi \sqrt{2\pi}
\]

C. \[
\frac{1}{x^2 + a^2} \quad (a > 0)
\]
\[
\frac{\pi e^{-|\xi|}}{\sqrt{2} a}
\]

D. \[
\begin{cases} 
x & \text{if } 0 < x \leq b, \\
2b - x & \text{if } b < x < 2b, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
-1 + 2e^{-ib\xi} - e^{-2ib\xi} \quad \xi^2 \sqrt{2\pi}
\]

E. \[
\begin{cases} 
e^{-ax} & \text{if } x > 0, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
\frac{1}{(a + i\xi)\sqrt{2\pi}}
\]

F. \[
\begin{cases} 
e^{ax} & \text{if } b < x < c, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
e^{(a-i\xi)c} - e^{(a-i\xi)b} \quad (a - i\xi) \sqrt{2\pi}
\]

G. \[
\begin{cases} 
e^{i\lambda x} & \text{if } -b < x < b, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
\sqrt{2} \sin(b(\xi - a)) \quad \xi - a \sqrt{\pi}
\]

H. \[
\begin{cases} 
e^{i\lambda x} & \text{if } c < x < d, \\
0 & \text{otherwise.} 
\end{cases}
\]
\[
e^{i\lambda(x-a)} - e^{i\lambda(x-a)} \quad i(\xi - a) \sqrt{2\pi}
\]

I. \[
e^{-ax^2} \quad (a > 0)
\]
\[
\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}
\]

J. \[
\frac{\sin(ax)}{x} \quad (a > 0)
\]
\[
\begin{cases} 
0 & \text{if } |\xi| \geq a, \\
\sqrt{\frac{\pi}{2}} & \text{if } |\xi| < a.
\end{cases}
\]
Fourier Series Convergence Theorems

Consider the eigenvalue problem

\[ X''(x) + \lambda X(x) = 0 \quad \text{in} \quad a < x < b \]

with any symmetric boundary conditions of the form

\[
\begin{aligned}
\alpha_1 f(a) + \beta_1 f(b) + \gamma_1 f'(a) + \delta_1 f'(b) &= 0 \\
\alpha_2 f(a) + \beta_2 f(b) + \gamma_2 f'(a) + \delta_2 f'(b) &= 0
\end{aligned}
\]

and let \( \Phi = \{ X_1, X_2, X_3, \ldots \} \) be the complete orthogonal set of eigenfunctions for (1)-(2). Let \( f \) be any absolutely integrable function defined on \( a \leq x \leq b \). Consider the Fourier series for \( f \) with respect to \( \Phi \):

\[
\sum_{n=1}^{\infty} A_n X_n(x)
\]

where

\[
A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \quad (n = 1, 2, 3, \ldots).
\]

**Theorem 2.** (Uniform Convergence) If

(i) \( f(x), f'(x), \text{ and } f''(x) \) exist and are continuous for \( a \leq x \leq b \) and

(ii) \( f \) satisfies the given symmetric boundary conditions,

then the Fourier series of \( f \) converges uniformly to \( f \) on \([a, b]\).

**Theorem 3.** (\( L^2 \) – Convergence) If

\[
\int_a^b |f(x)|^2 \, dx < \infty
\]

then the Fourier series of \( f \) converges to \( f \) in the mean-square sense in \((a, b)\).

**Theorem 4.** (Pointwise Convergence of Classical Fourier Series)

(i) If \( f \) is a continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) at \( x \) converges pointwise to \( f(x) \) in the open interval \( a < x < b \).

(ii) If \( f \) is a piecewise continuous function on \( a \leq x \leq b \) and \( f' \) is piecewise continuous on \( a \leq x \leq b \), then the classical Fourier series (full, sine, or cosine) converges pointwise at every point \( x \) in \((-\infty, \infty)\). The sum of the Fourier series is

\[
\sum_{n=1}^{\infty} A_n X_n(x) = \frac{f(x^+) + f(x^-)}{2}
\]

for all \( x \) in the open interval \((a, b)\).

**Theorem 4.** If \( f \) is a function of period \( 2l \) on the real line for which \( f \) and \( f' \) are piecewise continuous, then the classical full Fourier series converges to \( \frac{f(x^+) + f(x^-)}{2} \) for every real \( x \).
\[ (1 + x^2) u_x + 2xy^2 u_y = 0, \quad u(0, y) = y^3 \text{ if } y > 0. \]

The characteristic curves of \( a(x, y) u_x + b(x, y) u_y = 0 \) are given by
\[
\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \quad \text{In our case, this is } \frac{dy}{dx} = \frac{2xy^2}{1 + x^2} \Rightarrow \int \frac{1}{y^2} dy = \int \frac{2x dx}{1 + x^2}
\]
so \( \frac{-1}{y} = \ln(1 + x^2) + c \iff y = \frac{-1}{c + \ln(1 + x^2)}. \) Along any characteristic curve, the solution \( u = u(x, y) \) of the p.d.e. is constant. Along any such curve
\[
u(x, y(x)) = u(0, y(0)) = u(0, -\frac{1}{c}) = f(c).
\]
The general solution of the p.d.e. is thus
\[
u(x, y) = f \left( -\frac{1}{y} - \ln(1 + x^2) \right)
\]
where \( f \) is an arbitrary continuously differentiable function of a single real variable. We apply the auxiliary condition to obtain
\[ y^3 = u(0, y) = f \left( -\frac{1}{y} - \ln(1 + 0) \right) = f \left( -\frac{1}{y} \right) \quad \text{if } y > 0.
\]
Setting \( \zeta = -\frac{1}{y} \) we have \( f(\zeta) = \left( -\frac{1}{\zeta} \right)^3 = -\frac{1}{\zeta^3} \quad \text{for } \zeta < 0.
\]
Therefore
\[
u(x, y) = f \left( -\frac{1}{y} - \ln(1 + x^2) \right) = \frac{-1}{\left( -\frac{1}{y} - \ln(1 + x^2) \right)^3}
\]
or equivalently,
\[
u(x, y) = \frac{1}{\left[ \frac{1}{y} + \ln(1 + x^2) \right]^3} \quad \text{if } y > 0,
\]
\[
u(x, y) = \frac{y^3}{\left[ 1 + y \ln(1 + x^2) \right]^3} \quad \text{if } y > 0.
\]
(2) \[ u_{xx} - 2u_{xy} + u_{yy} + u_x - u_y = 4e^{y-x} \]

(a) \[ B^2 - 4AC = (-2)^2 - 4(1)(1) = 0 \] The p.d.e. is \textit{parabolic}.

(b) We "factor" the operators in the p.d.e. as follows.

\[
(\frac{\partial^2}{\partial x^2} - 2\frac{\partial}{\partial x} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2})u + (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})u = 4e^{y-x}
\]

\[
(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})u + (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})u = 4e^{y-x}.
\]

Let \[
\begin{cases}
\zeta = -(\beta x - \alpha y) = -(-x - y) = x + y,
\eta = \alpha x + \beta y = x - y.
\end{cases}
\]

The chain rule implies these differential operator expressions:

\[
\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta},
\]

\[
\frac{\partial}{\partial y} = \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta}.
\]

Therefore \[
\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}\right) - \left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta}\right) = 2 \frac{\partial}{\partial \eta} \]

so the p.d.e. can be rewritten in the \(\zeta\eta\)-coordinates as

\[
4 \frac{\partial u}{\partial \eta} + 2 \frac{\partial u}{\partial \eta} = 4e^{-\eta} \quad \text{[For an alternate solution see the]}
\]

bottom of the next page.

Letting \(\frac{\partial u}{\partial \eta} = v\), this reduces the p.d.e. to a first-order problem:

\[
4 \frac{\partial v}{\partial \eta} + 2v = 4e^{-\eta}
\]

\[
\frac{\partial v}{\partial \eta} + \frac{1}{2}v = e^{-\eta}.
\]

An integrating factor is \(\mu(v) = e^{-\frac{1}{4}v}\) since multiplying through
by the integrating factor yields
\[ e^{\frac{1}{2}\eta} \frac{dv}{d\eta} + \frac{1}{2} e^{\frac{1}{2}\eta} v = e^{-\frac{1}{2}\eta}. \]

But the left member is an exact expression:
\[ \frac{d}{d\eta}(e^{\frac{1}{2}\eta} v) = e^{\frac{1}{2}\eta} \frac{dv}{d\eta} + \frac{1}{2} e^{\frac{1}{2}\eta} v. \]

Therefore we have
\[ \frac{d}{d\eta}(e^{\frac{1}{2}\eta} v) = e^{-\frac{1}{2}\eta}. \]

and integration yields
\[ e^{\frac{1}{2}\eta} v = \int e^{-\frac{1}{2}\eta} d\eta = -2e^{-\frac{1}{2}\eta} + c_1(3) \]

so
\[ v = -2e^{-\eta} + c_1(3)e^{-\frac{1}{2}\eta}. \]

But \( v = \frac{du}{d\eta} \), so substituting in the equation above and integrating gives
\[ u = \int \frac{du}{d\eta} d\eta = \int (-2e^{-\eta} + c_1(3)e^{-\frac{1}{2}\eta}) d\eta \]
\[ u = 2e^{-\eta} - 2c_1(3)e^{-\frac{1}{2}\eta} + c_2(3) \]

As a function of \( x \) and \( y \), we have
\[ u(x,y) = 2e^{-x} + f(x+y)e^{-\frac{1}{2}(y-x)} + g(x+y) \]

where \( f \) and \( g \) are arbitrary twice continuously differentiable functions of a single real variable.

Alternate solution method starting at \((*) \)
\[ 4 \frac{d^2 u}{d\eta^2} + 2 \frac{du}{d\eta} = 4e^{-\eta}; \]

The general solution of the second order linear equation \((*) \) has the form
\[ u(x) = u_c(x) + u(x) \] where \( u_c \) is the general solution of the associated
homogeneous equation $4 \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial y} = 0$ and $u_p$ is any particular solution (i.e., without arbitrary constants) of the nonhomogeneous equation $(\ast)$. We look for exponential function solutions to $(\ast)$: $u = e^{ry}$ where $r$ is a constant. Then 
\[
\frac{\partial u}{\partial y} = re^{ry} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = r^2 e^{ry},
\]
so substituting in $(\ast)$ gives
\[
4r^2 e^{ry} + 2re^{ry} = 0 \Rightarrow 2e^{ry}(2r + 1) = 0.
\]

Therefore $r = 0$ or $r = -\frac{1}{2}$. Hence $u_c = c_1(\xi)e^{-\frac{y}{2}} + c_2(\xi)$ where $c_1$ and $c_2$ are arbitrary “constants” which may vary with the parameter $\xi$.

To find a particular solution of $(\ast)$, we use the method of undetermined coefficients. Looking at the right member of $(\ast)$, we expect a particular solution of the form $u_p = Ae^{-y}$ where $A$ is a constant to be determined. Then 
\[
\frac{\partial u_p}{\partial y} = -Ae^{-y} \quad \text{and} \quad \frac{\partial^2 u_p}{\partial y^2} = Ae^{-y}.
\]
We want to choose $A$ so that $u_p$ solves $(\ast)$:
\[
4 \frac{\partial^2 u_p}{\partial y^2} + 2 \frac{\partial u_p}{\partial y} = 4e^{-y}
\]
\[
4(Ae^{-y}) + 2(-Ae^{-y}) = 4e^{-y}
\]
\[
2Ae^{-y} = 4e^{-y}
\]
so we get a solution by choosing $A = 2$. Therefore
\[
u = u_c + u_p = c_1(\xi)e^{-\frac{y}{2}} + c_2(\xi) + 2e^{-y}.
\]

[The solution is finished as before.]
A p.d.e. and initial/boundary conditions governing the temperature \( u = u(x, y, z, t) \) are:

\[
\begin{align*}
(1) & \quad u_t - K \left( u_{xx} + u_{yy} + u_{zz} \right) = 0 & \text{if } 4 < x^2 + y^2 + z^2 < 100 \text{ and } 0 < t < \infty, \\
(2) & \quad \nabla u \cdot \vec{n} = 0 & \text{if } x^2 + y^2 + z^2 = 4 \text{ or } 100 \text{ and } t > 0, \\
(3) & \quad u(x, y, z, 0) = \frac{200}{\sqrt{x^2 + y^2 + z^2}} & \text{if } 4 \leq x^2 + y^2 + z^2 \leq 100.
\end{align*}
\]

\[
\frac{dH}{dt} = \frac{d}{dt} \iiint_{\Omega} c \rho u dV = \iiint_{\Omega} c \rho u_t dV = \iiint_{\Omega} k c \rho \nabla^2 u dV = \iiint_{\Omega} k c \rho \nabla \cdot (\nabla u) dV
\]

By Gauss' Divergence Theorem.

\[
\iiint_{\Omega} k c \rho \nabla u \cdot \vec{n} dS = \iint_{\partial \Omega} k c \rho \nabla u \cdot \vec{n} dS = 0.
\]

Therefore \( H(t) = H(0) = \text{constant for all } t > 0 \).

(c) Let \( U_\infty = \lim_{t \to \infty} u(x, y, z, t) \) be the (constant) steady-state temperature at all points in \( D \). Since \( H(0) = H(t) \) for all \( t > 0 \),

\[
H(0) = \lim_{t \to \infty} H(t) = \lim_{t \to \infty} \iiint_{\Omega} c \rho u dV = \iiint_{\Omega} c \rho (\lim_{t \to \infty} u) dV = \iiint_{\Omega} c \rho U_\infty dV
\]

\[
= c \rho U_\infty \text{vol}(\Omega) = c \rho U_\infty \left( \frac{4}{3} \pi (10^3) - \frac{4}{3} \pi (2^3) \right) = c \rho U_\infty \frac{4}{3} \pi (992).
\]

But \( H(0) = \iiint_{\Omega} c \rho u(x, y, z, 0) dV = \iiint_{\Omega} \frac{200}{\sqrt{x^2 + y^2 + z^2}} dV = \int_0^{2\pi} \int_0^1 \int_0^\pi 200 r^2 \sin \phi \, dr \, d\phi \, d\theta \)

\[
= c \rho \left( \int_0^{2\pi} d\theta \right) \left( \int_0^\pi \sin \phi d\phi \right) \left( \int_0^{10} 200 r^2 dr \right) = c \rho (2\pi)(2)(100)(\frac{10}{2}) = c \rho (400\pi)(96).
\]
Thus

\[ c \rho (400 \pi)(96) = c \rho U_\infty \frac{4}{3} \pi (992) \]

\[ \Rightarrow \quad U_\infty = \frac{\frac{\rho}{2}}{\frac{992}{2}} \cdot 3 \quad \frac{900}{31} \] (This is approximately 29.)
\[
\begin{align*}
\#4 & \quad \begin{cases}
    u_t - u_{xx} = 0 & \text{if } -\infty < x < \infty, \quad 0 < t < \infty, \\
    u(x,0) = x^3 & \text{if } -\infty < x < \infty.
\end{cases}
\end{align*}
\]

By the formula for solution to the heat equation in the upper half-plane,
\[
\begin{align*}
    u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} q(y) \, dy,
\end{align*}
\]

with \( k=1 \) and \( q(y) = y^3 \), we have
\[
\begin{align*}
    (*) & \quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^3 \, dy.
\end{align*}
\]

(Actually, the formula is only guaranteed to yield a solution provided \( q \) is bounded and continuous on the entire real line. Since our function \( q(y) = y^3 \) is not bounded, we will need to check our answer.)

Let \( p = \frac{y-x}{\sqrt{4t}} \) in the integral in (\*). Then \( dp = \frac{dy}{\sqrt{4t}} \) and \( \sqrt{4t} p + x = y \) so
\[
\begin{align*}
    u(x,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left( \frac{1}{\sqrt{4t}} (p + x) \right)^3 dp \\
    &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left\{ \left( \frac{1}{\sqrt{4t}} p + x \right)^3 + 3 \left( \frac{1}{\sqrt{4t}} p \right)^2 x + 3 \left( \frac{1}{\sqrt{4t}} p \right) x^2 + x^3 \right\} dp \\
    &= \frac{4t}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^3 dp + 12t \int_{-\infty}^{\infty} e^{-p^2} p dp + 3 \sqrt{\frac{4t}{\pi}} x \int_{-\infty}^{\infty} e^{-p^2} p dp + \frac{x^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp
\end{align*}
\]

But \( \int_{-\infty}^{\infty} e^{-p^2} p^3 dp = 0 = \int_{-\infty}^{\infty} e^{-p^2} p dp \) since the integrands are odd functions,
\[
\begin{align*}
    \int_{-\infty}^{\infty} e^{-p^2} dp &= \sqrt{\pi}, \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-p^2} p^2 dp = \frac{\sqrt{\pi}}{2} \quad (\text{given identity}) \quad \text{so}
\end{align*}
\]

\[
\boxed{u(x,t) = x^3 + 6tx}
\]

Check: \( u_t - u_{xx} = 6x - 6x = 0, \quad u(x,0) = x^3 \).
\[ \begin{aligned} u_t - u_{xx} + 2tu &= f(x,t) \quad \text{if} \quad -\infty < x < \infty, \quad 0 < t < \infty, \\ u(x,0) &= 0 \quad \text{if} \quad -\infty < x < \infty. \end{aligned} \]

Suppose \( u = u(x,t) \) is a solution to (1)-(2) and take the Fourier transform (with respect to \( x \)) of (1):

\[ \mathcal{F}\{ u_t - u_{xx} + 2tu \}(\xi) = \mathcal{F}\{ f(\cdot,t) \}(\xi). \]

Then

\[ \frac{2}{\partial t} \mathcal{F}\{ u \}(\xi) - (\xi^2) \mathcal{F}\{ u \}(\xi) + 2t \mathcal{F}\{ u \}(\xi) = \hat{f}(\xi,t). \]

\( \hat{f}(\xi,t) \)

This is a first order linear differential equation in the independent variable \( t \).

An integrating factor is

\[ \mu(t) = e^{\int (\xi^2 + 2t) dt} = e^{\frac{\xi^2 + t^2}{2}}. \]

Multiplying (2) by the integrating factor and combining terms gives

\[ \frac{\partial}{\partial t} \left( e^{\frac{\xi^2 + t^2}{2}} \mathcal{F}\{ u \}(\xi) \right) = e^{\frac{\xi^2 + t^2}{2}} \mathcal{F}\{ f(\cdot,t) \}(\xi) + (\xi^2 + 2t)e^{\frac{\xi^2 + t^2}{2}} \mathcal{F}\{ u \}(\xi) = e^{\frac{\xi^2 + t^2}{2}} \hat{f}(\xi,t). \]

Integrating yields

\[ e^{\frac{\xi^2 + t^2}{2}} \mathcal{F}\{ u \}(\xi) = \int_0^t e^{\frac{\xi^2 + \tau^2}{2}} \hat{f}(\xi,\tau) d\tau + C(\xi). \]

Applying (2) to the equation above leads to

\[ 0 = e^{\frac{\xi^2 + t^2}{2}} \mathcal{F}\{ u \}(\xi) \bigg|_{t=0}^{t=0} = \left( \int_0^t e^{\frac{\xi^2 + \tau^2}{2}} \hat{f}(\xi,\tau) d\tau + C(\xi) \right) \bigg|_{t=0}^{t=0} = C(\xi) \]

Thus

\[ \mathcal{F}\{ u \}(\xi) = e^{-\frac{\xi^2 + t^2}{2}} \int_0^t e^{\frac{\xi^2 + \tau^2}{2}} \hat{f}(\xi,\tau) d\tau = \int_0^t e^{-\frac{(\xi^2 + \tau^2 - 2t\tau)}{2a}} f(\xi,\tau) d\tau \]

Applying entry 1 in the Fourier transform table:

\[ \mathcal{F}\{ e^{-ac^2} \}(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{2a}}. \]
with \( t - \tau = \frac{1}{4a} \), or equivalently \( a = \frac{1}{4(t - \tau)} \), gives

\[
\mathcal{F} \left( \frac{1}{\sqrt{2(\pi)}(t - \tau)} e^{-\frac{(\zeta - t)^2}{4(t - \tau)}} \right)(\xi) = e^{-\frac{\xi^2}{(t - \tau)}}
\]

(\text{III})

Substituting this in (\text{III}) produces

\[
\mathcal{F}(u)(\xi) = \int_0^t e^\frac{\tau^2 - \xi^2}{4\pi(t - \tau)} \mathcal{F} \left( \frac{1}{\sqrt{2(\pi)}(t - \tau)} e^{-\frac{(\zeta - t)^2}{4(t - \tau)}} \right)(\xi) \mathcal{F}(f(\cdot, \tau))(\xi) d\tau.
\]

(10)

The convolution property of Fourier transforms, \( \mathcal{F}(f)(\xi) \mathcal{F}(g)(\xi) = \frac{1}{\sqrt{4\pi}} \mathcal{F}(f * g)(\xi) \), then implies

\[
\mathcal{F}(u)(\xi) = \int_0^t e^\frac{\tau^2 - \xi^2}{4\pi(t - \tau)} \mathcal{F} \left( \frac{1}{\sqrt{2(\pi)}(t - \tau)} e^{-\frac{(\zeta - t)^2}{4(t - \tau)}} * f(\cdot, \tau) \right)(\xi) d\tau
\]

(10)

Interchanging the order of the integrations — recall that the Fourier transform is an integral with respect to \( x \) over the interval \(-\infty < x < \infty\) — yields

\[
\mathcal{F}(u)(\xi) = \mathcal{F} \left( \int_0^t e^\frac{\tau^2 - \xi^2}{4\pi(t - \tau)} \mathcal{F} \left( \frac{1}{\sqrt{2(\pi)}(t - \tau)} e^{-\frac{(\zeta - t)^2}{4(t - \tau)}} \right) * f(\cdot, \tau) \right)(\xi)
\]

(20)

and consequently the inversion theorem gives

\[
u(x, t) = \int_0^t e^\frac{\tau^2 - \xi^2}{4\pi(t - \tau)} \mathcal{F} \left( \frac{1}{\sqrt{2(\pi)}(t - \tau)} e^{-\frac{(\zeta - t)^2}{4(t - \tau)}} \right) * f(\cdot, \tau) \right)(\xi) d\tau
\]

(21)

\[
= \int_0^t e^\frac{\tau^2 - \xi^2}{4\pi(t - \tau)} \int_{-\infty}^{\infty} e^\frac{-\xi^2}{4(t - \tau)} f(y, \tau) \, dy \, d\tau
\]

(15)
(15) (a) The Fourier cosine series for a function \( \varphi \) on \((0, \ell)\) is

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi x}{\ell} \right)
\]

where

\[
a_n = \frac{\left< \varphi, \cos(n \pi c/\ell) \right>}{\left< \cos(n \pi c/\ell), \cos(n \pi c/\ell) \right>}
= \frac{2}{\ell} \int_0^\ell \varphi(x) \cos\left( \frac{n \pi x}{\ell} \right) dx \quad (n \geq 1)
\]

and

\[
a_0 = \frac{\left< \varphi, 1 \right>}{\left< 1, 1 \right>}
= \frac{1}{\ell} \int_0^\ell \varphi(x) dx.
\]

In our case \( \ell = 1 \) so

\[
a_0 = \int_0^1 \varphi(x) dx = \int_0^1 (2x^2 - x^4) dx = \left[ \frac{2x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{10}{3} - \frac{3}{15} = \frac{7}{15}.
\]

Note that

\[
\varphi'(x) = 4x - 4x^3
\]

so \( \varphi'(0) = 0 = \varphi'(1) \).

\[
\varphi''(x) = 4 - 12x^2
\]

\[
\varphi''(x) = -24x
\]

\[
\varphi^{(4)}(x) = -24
\]

since \( \varphi^{(4)} \) is constant and \( \int_0^\ell \cos(n \pi x) dx = 0 \)

\[
\frac{2}{(n \pi)^3} \left[ -\frac{\varphi''(x) \cos(n \pi x)}{n \pi} \right]_0^1 + \frac{1}{(n \pi)^3} \int_0^1 \frac{\varphi^{(4)}(x) \cos(n \pi x) dx}{n \pi}
\]

\[
= \frac{2}{(n \pi)^3} \left( 24 \cos(n \pi x) \right) \bigg|_0^1
\]

\[
= \frac{48(-1)^n}{(n \pi)^3} \quad \text{if } n \geq 1.
\]
Therefore the Fourier cosine series for \( f \) on \([0,1]\) is

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \frac{7}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^4}.
\]

(b) \( \mathcal{E} = \{ 1, \cos(n\pi x), \cos(2n\pi x), \cos(3n\pi x), \ldots \} \) is the complete orthogonal set of eigenfunctions for

(1) \( \mathcal{E}''(x) + \lambda \mathcal{E}(x) = 0 \) in \( 0 < x < 1 \)

with the symmetric (Neumann) boundary conditions

(2) \( \mathcal{E}(0) = 0 \) and \( \mathcal{E}'(1) = 0 \).

Note: \( f(x) = 2x^2 - x^4 \)

\[
\begin{align*}
\phi'(x) &= 4x - 4x^3 \\
\phi''(x) &= 4 - 12x
\end{align*}
\]

These derivatives exist and are continuous for \( 0 \leq x \leq 1 \).

(4) \( \phi'(0) = 0 = \phi'(1) \) so \( f \) satisfies the boundary conditions (2) as well. Thus Theorem 2 implies that the Fourier cosine series of \( f \) converges uniformly to \( f \) on \([0,1]\).
\[ u_{tt} - u_{xx} = 0 \quad \text{if } 0 < x < 1, \quad 0 < t < \infty, \]
\[ u_x(0,t) = 0 \overset{\circ}{=} u_x(1,t) \quad \text{if } t \geq 0, \]
\[ u_t(x,0) = 0 \quad \text{if } 0 \leq x \leq 1, \]
\[ u(x,0) = \begin{cases} 2x^2 - x^4 & \text{if } 0 \leq x \leq 1. \end{cases} \]

(a) We use separation of variables. We seek nontrivial solutions of \(1-2-3-4\) of the form \(u(x,t) = \Xi(x)T(t)\). Substituting in \(1\) gives
\[
\Xi(x)T''(t) - \Xi''(x)T(t) = 0 \quad \Rightarrow \quad -\frac{\Xi''(x)}{\Xi(x)} = \frac{T''(t)}{T(t)} = \text{constant} = \lambda.
\]
Substituting in \(2-3\) yields
\[
\Xi'(0)T(t) = 0 \quad \text{and} \quad \Xi'(1)T(t) = 0 \quad \text{if } t \geq 0.
\]
In order to have a nontrivial solution, it follows that \(\Xi'(0) = \Xi'(1) = 0\).
Similarly, substituting in \(4\) gives
\[
\Xi(x)T'(x) = 0 \quad \text{if } 0 \leq x \leq 1.
\]
and hence \(T'(0) = 0\) in order to have a nontrivial solution. Collecting these facts, we have
\[
\begin{cases}
\Xi''(x) + \lambda \Xi(x) = 0, \quad \Xi'(0) = 0 = \Xi'(1), \\
T''(t) + \lambda T(t) = 0, \quad T'(0) = 0.
\end{cases}
\]
(Note that \(6-7-8\) is an eigenvalue problem.) In the textbook, section 4.2, it is shown that the eigenvalues and eigenfunctions of \(T = -\frac{d^2}{dx^2}\) with Neumann boundary conditions \(7-8\) are, respectively,
\[
\lambda_n = (n\pi)^2 \quad \text{and} \quad \Xi_n(x) = \cos(n\pi x), \quad (n = 0, 1, 2, \ldots).
\]
Substituting \(\lambda = \lambda_n = (n\pi)^2\) in \(9\), we find its general solution is
\[
T_n(t) = \alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t).
\]
Applying \(10\), we see that \(T_n(t) = \cos(n\pi t)\) for \(n = 0, 1, 2, 3, \ldots\) (up to a constant multiple, anyway).
Consequently, \( u_n(x,t) = \sum_n T_n(t) \cos(n \pi x) \) \((n = 0, 1, 2, \ldots)\)
solves (1)-(2)-(3)-(4). The superposition principle then gives a formal solution
\[
u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \pi x) \cos(n \pi t)\]
to (1)-(2)-(3)-(4) for any choice of constants \(a_0, a_1, a_2, \ldots\). Applying (5) yields
\[
x^2 - x^4 = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \pi x) \quad \text{if} \quad 0 \leq x \leq 1.
\]
By problem (6), we may take
\[
a_0 = \frac{7}{15} \quad \text{and} \quad a_n = \frac{48(-1)^n}{n^4 \pi^4} \quad \text{if} \quad n \geq 1.
\]
Hence a solution to (1)-(2)-(3)-(4)-(5) is
\[
u(x,t) = \frac{7}{15} + \frac{48}{\pi^4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n \pi x) \cos(n \pi t)}{n^4}}.
\]

(b) The solution to (1)-(2)-(3)-(4)-(5) obtained above is unique. To see this, suppose that \( u = v(x,t) \) were another solution and define \( w(x,t) = u(x,t) - v(x,t) \). Then \( w \) solves
\[
\begin{cases}
w_{tt} - w_{xx} = 0 \quad \text{if} \quad 0 < x < 1, 0 < t < \infty, \\
w_x(0,t) = 0 = w_x(1,t) \quad \text{if} \quad t \geq 0, \\
w(x,0) = 0 = w_t(x,0) \quad \text{if} \quad 0 \leq x \leq 1.
\end{cases}
\]
Let
\[
E(t) = \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] dx \quad (t \geq 0)
\]
be the energy function of \( w \) on \([0, \infty)\). Then
\[
\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_x^2(x,t)] dx \right)
\]
\[
\frac{dE}{dt} = \frac{1}{2} \int_0^1 \frac{2}{\partial t} \left[ w_t^2(x,t) + w_x^2(x,t) \right] dx = \int_0^1 \left[ w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx.
\]

Using (11) to substitute for \( w_{tt} \) in the integrand of the left member of the equation above and integrating by parts the second term in the integrand with \( U = w_x(x,t) \) and \( dV = w_{xt}(x,t) dx \), we find

\[
\frac{dE}{dt} = \int_0^1 \left[ w_t(x,t) w_{xx}(x,t) dx + w_x(x,t) w_t(x,t) \right] - \int_0^1 w_t(x,t) w_{xx}(x,t) dx
\]

\[x=0\]

\[= w_x(1,t) w_t(1,t) - w_x(0,t) w_t(0,t)\]

\[= 0\]

by (12) and (13). Therefore the energy of \( W \) is a constant function on \([0,\infty)\) so

\[0 \leq E(t) = E(0) = \int_0^1 \left[ w_t^2(x,0) + w_x^2(x,0) \right] dx = 0\]

by (15) and differentiation of (14): \( W(x,0) = 0 \) if \( 0 \leq x \leq 1 \) implies \( w_x(x,0) = 0 \) if \( 0 \leq x \leq 1 \). Consequently, the vanishing theorem yields that \( w_t^2(x,t) = 0 = w_x^2(x,t) \) for all \( 0 \leq x \leq 1 \) and each \( t \geq 0 \) so

\[W(x,t) = \text{constant in } 0 \leq x \leq 1, 0 \leq t < \infty. \text{ But (14) then gives } W(x,t) = 0 \]

if \( 0 \leq x \leq 1, 0 \leq t < \infty. \) That is, \( u(x,t) = v(x,t) \) for all \( 0 \leq x \leq 1 \) and \( 0 \leq t < \infty. \)
(a) The steady-state temperature distribution $u = u(x, y, z)$ of the material in the spherical shell satisfies

$$
\begin{align*}
\nabla^2 u &= 0 \quad \text{if} \quad 1 < x^2 + y^2 + z^2 < 4, \\
\frac{\partial u}{\partial n} &= 100 \quad \text{if} \quad x^2 + y^2 + z^2 = 1, \\
\n\nabla u \cdot \vec{n} &= -\kappa \quad \text{if} \quad x^2 + y^2 + z^2 = 4.
\end{align*}
$$

Since the region, partial differential equation, and boundary conditions are invariant under rotations about the origin, we expect that any solution to (1) - (3) at each point depends only on the distance of that point from the origin. I.e., in spherical coordinates we expect $u = u(r)$, independent of $\theta$ and $\phi$. Then $\partial u/\partial \phi = 0$ and $\partial^2 u/\partial \theta^2 = 0$ so (1) becomes

$$0 = \nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right)$$

Multiplying by $r^2$ and integrating yields

$$c_1 = r^2 \frac{\partial u}{\partial r}.$$  \tag{**}

In spherical coordinates, (3) becomes $\frac{\partial u}{\partial r} = -\kappa$ if $r = 2$. Consequently $c_1 = r^2 \frac{\partial u}{\partial r} \bigg|_{r=2} = -4\kappa$. Substituting in (**) gives

$$r = 2 \quad -4\kappa = r^2 \frac{\partial u}{\partial r}.$$  

Rearranging and integrating yields

$$u = \int \frac{\partial u}{\partial r} \, dr = \int -\frac{4\kappa}{r^2} \, dr = -\frac{4\kappa}{r} + c_2.$$  

Applying (2) produces

$$100 = u \bigg|_{r=1} = \left( \frac{4\kappa}{r} + c_2 \right) \bigg|_{r=1} = 4\kappa + c_2,$$

so $c_2 = 100 - 4\kappa$ and

$$u(r) = \frac{4\kappa}{r} + 100 - 4\kappa \quad \text{if} \quad 1 \leq r \leq 2.$$

(This solution is unique. See the note on the last page of these solutions.)
(b) Since $u(r) = \frac{4K}{r} + 100 + 4K$ is harmonic in the region $1 < r < 2$ and continuous on its closure $1 \leq r \leq 2$, the maximum/minimum principle states the hottest and coldest temperatures must occur on the boundaries of the region. Hence
\[
\max u = u(1) = \frac{4K}{1} + 100 - 4K = 100
\]
and
\[
\min u = u(2) = \frac{4K}{2} + 100 - 4K = 100 - 2K.
\]

(c) If we choose $K = 40$, then the work in part (b) shows that $u(2) = 100 - 2K = 100 - 80 = 20^\circ$ Celsius. Therefore it is possible to choose $K$ so the temperature on the outer boundary is 20 degrees Celsius.
Note: The solution to \( 1-2-3 \) is unique. To see this, suppose that \( u = u(x,y,z) \) and \( v = v(x,y,z) \) were solutions to \( 1-2-3 \) and let \( w(x,y,z) = u(x,y,z) - v(x,y,z) \). Then \( w \) solves

\[
\begin{align*}
\Delta^2 w &= 0 & \text{if } 1 < r < 2, \\
w &= 0 & \text{if } r = 1, \\
\nabla w \cdot \vec{n} &= 0 & \text{if } r = 2.
\end{align*}
\]

Let \( E = \iiint_D |\nabla w|^2 \, dV \) where \( D = \{(x,y,z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 4 \} \).

Recall the identity \( \nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g \). Rearranging leads to \( \nabla f \cdot \nabla g = \nabla \cdot (f \nabla g) - f \nabla^2 g \) and setting \( f = g = w \) gives

\[
|\nabla w|^2 = \nabla w \cdot \nabla w = \nabla \cdot (w \nabla w) - w \nabla^2 w.
\]

Substituting in the expression for \( E \) and using \( 1 \) produces

\[
E = \iiint_D \left[ \nabla \cdot (w \nabla w) - w \nabla^2 w \right] \, dV = \iiint_D \nabla \cdot (w \nabla w) \, dV.
\]

Applying Gauss' divergence theorem yields

\[
E = \iint_{\partial D} w \nabla w \cdot \vec{n} \, dS.
\]

On the \( r = 1 \) portion of \( \partial D \), \( 2 \) implies \( w \nabla w \cdot \vec{n} = 0 \). On the \( r = 2 \) portion of \( \partial D \), \( 3 \) implies \( w \nabla w \cdot \vec{n} = 0 \). Hence \( E = 0 \) and the vanishing theorem implies \( |\nabla w| = 0 \) in \( D \). Consequently \( w = \text{constant} \) in \( D \). Condition \( 2 \) then yields \( w = 0 \) in \( D \); i.e. \( u(x,y,z) = v(x,y,z) \) for all \( (x,y,z) \) in \( D \).
Math 325
Final Exam
Summer 2013

number taking exam = 8
median = 139
mean = 134.9
standard deviation = 28.8

Distribution of Scores

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