Mathematics 5325 Exam III Name: Dr. Grow

Summer 2015

1.(33 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem:

$$u_t - u_{xx} + 2tu \overset{①}{=} f(x,t) \text{ if } -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) \overset{②}{=} 0 \text{ if } -\infty < x < \infty.$$

Let $u = u(x,t)$ be a solution to ①-② and take the Fourier transform of ① with respect to x, holding t fixed:

$$\mathcal{F}\left(u_t(\cdot,t) - u_{xx}(\cdot,t) + 2tu(\cdot,t)\right)(\xi) = \mathcal{F}\left(f(\cdot,t)\right)(\xi).$$

Then the properties of Fourier transforms gives

$$\mathcal{F}\left(u_t(\cdot,t)\right)(\xi) - (i\xi)^2\mathcal{F}\left(u(\cdot,t)\right)(\xi) + 2t\mathcal{F}\left(u(\cdot,t)\right)(\xi) = \hat{f}(\xi,t).$$

Interchanging integration with respect to x (in the Fourier transform) and differentiation with respect to t gives

$$(*) \qquad \frac{\partial}{\partial t}\mathcal{F}\left(u(\cdot,t)\right)(\xi) + (\xi^2+2t)\mathcal{F}\left(u(\cdot,t)\right)(\xi) = \hat{f}(\xi,t).$$

This is a first order, linear differential equation in the independent variable t with parameter ξ. An integrating factor is

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int(\xi^2+2t)\,dt} = e^{t^2+\xi^2 t + \cancel{c}^{\,0}}.$$

Multiplying $(*)$ by the integrating factor and combining terms gives

$$\frac{\partial}{\partial t}\left(e^{t^2+\xi^2 t}\mathcal{F}\left(u(\cdot,t)\right)(\xi)\right) = e^{t^2+\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}\left(u(\cdot,t)\right)(\xi) + (2t+\xi^2)\mathcal{F}\left(u(\cdot,t)\right)(\xi) = e^{t^2+\xi^2 t}\hat{f}(\xi,t).$$

Integrating yields

$$e^{t^2+\xi^2 t}\mathcal{F}\left(u(\cdot,t)\right)(\xi) = \int_0^t e^{\tau^2+\xi^2\tau}\hat{f}(\xi,\tau)\,d\tau + c(\xi).$$

Applying the initial condition ② to the previous equation leads to

$$0 = e^{t^2+\xi^2 t}\mathcal{F}\left(u(\cdot,t)\right)(\xi)\Big|_{t=0} = \left(\int_0^t e^{\tau^2+\xi^2\tau}\hat{f}(\xi,\tau)\,d\tau + c(\xi)\right)\Bigg|_{t=0} = c(\xi).$$

Therefore

$$(**) \qquad \mathcal{F}\left(u(\cdot,t)\right)(\xi) = e^{-t^2-\xi^2 t}\int_0^t e^{\tau^2+\xi^2\tau}\hat{f}(\xi,\tau)\,d\tau = \int_0^t e^{\tau^2-t^2}\cdot e^{-\xi^2(t-\tau)}\hat{f}(\xi,\tau)\,d\tau.$$

Apply entry I in the Fourier transform table, $\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}$, with $t-\tau = \frac{1}{4a}$, or equivalently $a = \frac{1}{4(t-\tau)}$, to give

$$\mathcal{F}\left(\frac{1}{\sqrt{2(t-\tau)}} e^{-\frac{(\cdot)^2}{4(t-\tau)}} \right)(\xi) = e^{-\xi^2(t-\tau)}.$$

Substituting this in (**) produces

$$\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t e^{\tau^2-t^2} \mathcal{F}\left(\frac{1}{\sqrt{2(t-\tau)}} e^{\frac{-(\cdot)^2}{4(t-\tau)}} \right)(\xi) \mathcal{F}(f(\cdot,\tau))(\xi)\, d\tau.$$

The convolution property of Fourier transforms, $\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f*g)(\xi)$ then implies

$$\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t e^{\tau^2-t^2} \mathcal{F}\left(\frac{1}{\sqrt{4\pi(t-\tau)}} e^{\frac{-(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau) \right)(\xi)\, d\tau$$

$$= \int_0^t \mathcal{F}\left(\frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau) \right)(\xi)\, d\tau.$$

Interchanging the order of integrations (recall that the Fourier transform is an integral over $-\infty < x < \infty$) yields

$$\mathcal{F}(u(\cdot,t))(\xi) = \mathcal{F}\left(\int_0^t \frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau)\, d\tau \right)(\xi).$$

Apply the uniqueness theorem for Fourier transforms to obtain

$$u(x,t) = \int_0^t \left(\frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-(\cdot)^2}{4(t-\tau)}} * f(\cdot,\tau) \right)(x)\, d\tau.$$

Writing the convolution product as an integral, $(g*h)(x) = \int_{-\infty}^{\infty} g(x-y)h(y)\,dy$, we have

$$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{\tau^2-t^2}}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4(t-\tau)}} f(y,\tau)\, dy\, d\tau.}$$

Equivalently, interchanging the order of integrations, we may express the solution as

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} \int_0^t \frac{e^{\frac{-(x-y)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}} \cdot e^{\tau^2-t^2} f(y,\tau)\, d\tau\, dy}.$$

2.(33 pts.) Consider the eigenvalue problem

(*) $$X''(x)+\lambda X(x)\overset{①}{=}0, \quad X'(0)\overset{②}{=}0\overset{③}{=}X'(1)-X(1).$$

(a) Find an equation that characterizes the positive eigenvalues of (*).

(b) Show graphically from the equation in part (a) that there is an infinite sequence of positive eigenvalues for (*).

(c) Approximate to eight decimal places the first three positive eigenvalues of (*).

(d) Is zero an eigenvalue of (*)? Support your answer.

(e) Are there any negative eigenvalues of (*)? Support your answer.

(f) Write the eigenfunction(s) corresponding to each eigenvalue of (*).

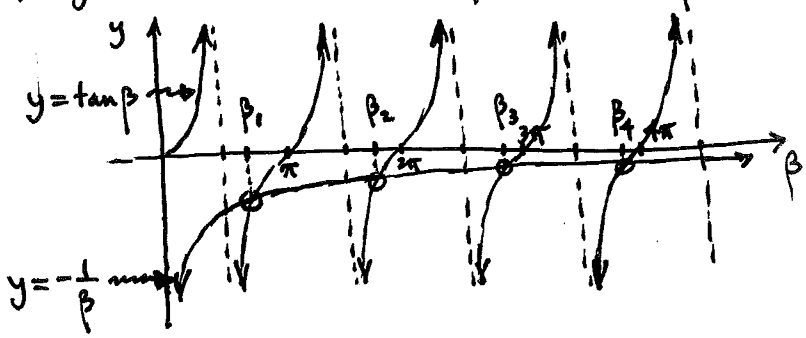
6 (a) Suppose $\lambda > 0$ in (*), say $\lambda = \beta^2$ where $\beta > 0$. Then ① becomes $X''(x)+\beta^2 X(x)=0$. The general solution of this DE is $X(x) = c_1\cos(\beta x)+c_2\sin(\beta x)$. Then $X'(x) = -\beta c_1\sin(\beta x)+\beta c_2\cos(\beta x)$ so applying the BCs gives

$$0 \overset{②}{=} X'(0) = -\beta c_1\cdot 0+\beta c_2\cdot 1 \Rightarrow c_2 = 0;$$

$$0 \overset{③}{=} X'(1)-X(1) = -\beta c_1\sin(\beta)+\beta \cancel{c_2}\cos(\beta) - (c_1\cos(\beta)+\cancel{c_2}\sin(\beta)) = -c_1(\beta\sin(\beta)+\cos(\beta)).$$

Therefore, in order to have a nontrivial solution to ①-②-③ when $\lambda=\beta^2>0$, we must have $c_1 \neq 0$ and $\beta\sin\beta + \cos\beta = 0$, or equivalently, $\boxed{\tan\beta = -\frac{1}{\beta}}$.

5 (b) Graphing the two curves $y=\tan\beta$ and $y=-\frac{1}{\beta}$ for $\beta>0$, we have:



Clearly, there are an infinite number of intersection points of $y=-\frac{1}{\beta}$ with the branches of $y=\tan\beta$ when $\beta>0$. Therefore there is an infinite sequence of positive eigenvalues $\lambda_n = \beta_n^2$ $(n=1,2,3,\ldots)$ satisfying $\lim_{n\to\infty}\left(\lambda_n - (n\pi)^2\right) = 0$.

6 (c) In order to approximate the first three positive solutions to $0=f(x)=\tan(x)+\frac{1}{x}$, we used "solve equation" in the NUMSLV menu on the HP 49G calculator with

the indicated seeds.

n	seed	β_n approximation	$\lambda_n = \beta_n^2$ approximation
1	3	2.79838604578	$\boxed{7.83096446122 \doteq \lambda_1}$
2	6	6.12125046690	$\boxed{37.4697072785 \doteq \lambda_2}$
3	9	9.31786646179	$\boxed{86.8226353998 \doteq \lambda_3}$

5 (d) When $\lambda = 0$ in ①, the DE is $X''(x) = 0$ with general solution $X(x) = c_1 x + c_2$.
Then $X'(x) = c_1$, so applying the BCs gives

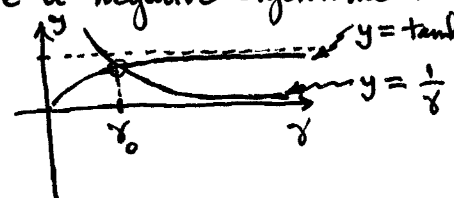
$$0 \overset{②}{=} X'(0) = c_1,$$
$$0 \overset{③}{=} X'(1) - X(1) = c_1 - (c_1 + c_2) = -c_2.$$

Therefore $X(x) \equiv 0$ so there are no nontrivial solutions to ①-②-③ when $\lambda = 0$.
That is, $\boxed{\text{zero is not an eigenvalue of ①-②-③.}}$

6 (e) Suppose $\lambda < 0$ in (*), say $\lambda = -\gamma^2$ where $\gamma > 0$. Then ① becomes $X''(x) - \gamma^2 X(x) = 0$.
The general solution of this DE is $X(x) = c_1 \cosh(\gamma x) + c_2 \sinh(\gamma x)$. Then $X'(x) = \gamma c_1 \sinh(\gamma x) + \gamma c_2 \cosh(\gamma x)$
so applying the BCs gives

$$0 \overset{②}{=} X'(0) = \gamma c_1 \cdot 0 + \gamma c_2 \cdot 1 = \gamma c_2 \Rightarrow c_2 = 0;$$
$$0 \overset{③}{=} X'(1) - X(1) = \gamma c_1 \sinh(\gamma) + \gamma \cancel{c_2}^{0} \cosh(\gamma) - (c_1 \cosh(\gamma) + \cancel{c_2}^{0} \sinh(\gamma)) = c_1(\gamma \sinh(\gamma) - \cosh(\gamma)).$$

Therefore, in order to have a negative eigenvalue we must have $c_1 \neq 0$ and $\gamma \sinh(\gamma) = \cosh(\gamma)$,
i.e. $\tanh(\gamma) = 1/\gamma$.



From the graphs, it is apparent there is one intersection point of $y = \tanh(\gamma)$ and $y = \frac{1}{\gamma}$ if $\gamma > 0$.
Hence $\boxed{\text{there is exactly one negative eigenvalue } \lambda_0 = -\gamma_0^2 \text{ of ①-②-③.}}$

5 (f) The eigenfunctions of ①-②-③ are:

$$\boxed{\begin{aligned} &X_n(x) = \cos(\beta_n x) \quad \text{for } \lambda_n = \beta_n^2 \quad (n = 1, 2, 3, \ldots) \\ &X_0(x) = \cosh(\gamma_0 x) \quad \text{for } \lambda_0 = -\gamma_0^2. \end{aligned}}$$

3.(34 pts.) (a) Find a solution to the damped wave equation

(1) $$u_{tt} - u_{xx} + 2u_t = 0 \ \text{ in } \ 0 < x < \pi, \ 0 < t < \infty,$$

satisfying

(2)-(3) $$u_x(0,t) = 0 = u_x(\pi,t) \ \text{ for } \ t \ge 0,$$

(4) $$u(x,0) = 0 \ \text{ for } \ 0 \le x \le \pi,$$

and

(5) $$u_t(x,0) = \frac{1}{2} + \frac{1}{2}\cos(2x) \ \text{ for } \ 0 \le x \le \pi.$$

(You may find useful the facts that $y'' + 2y' + n^2 y = 0$ has general solution $y_0(t) = c_1 + c_2 e^{-2t}$ if $n = 0$, $y_1(t) = c_1 e^{-t} + c_2 t e^{-t}$ if $n = 1$, and $y_n(t) = e^{-t}\left(c_1 \cos\left(t\sqrt{n^2-1}\right) + c_2 \sin\left(t\sqrt{n^2-1}\right)\right)$ if $n = 2,3,4,\ldots$)

(b) Show that the energy $E(t) = \int_0^\pi \left[\frac{1}{2}u_t^2(x,t) + \frac{1}{2}u_x^2(x,t)\right]dx$ of a solution to (1)-(2)-(3) is a decreasing function for $t \ge 0$.

(c) Show that the solution to (1)-(2)-(3)-(4)-(5) is unique.

(22) (a) We use separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem, i.e. (1)-(2)-(3)-(4), of the form $u(x,t) = X(x)T(t)$. Substituting in (1) yields $X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$ so

(*) $$\frac{-X''(x)}{X(x)} = \frac{-(T''(t) + 2T'(t))}{T(t)} = \text{constant} = \lambda.$$

Substituting $u(x,t) = X(x)T(t)$ in (2), (3), and (4) gives $X'(0)T(t) = 0 = X'(\pi)T(t)$ for $t \ge 0$ and $X(x)T(0) = 0$ for $0 \le x \le \pi$. Thus nontriviality of u implies

4 (6)-(7)-(8) $$X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(\pi),$$

6 (9)-(10) $$T''(t) + 2T'(t) + \lambda T(t) = 0, \quad T(0) = 0.$$

By Section 4.2, the eigenvalues and eigenfunctions of the Neumann problem (6)-(7)-(8)

8 are $\lambda_n = n^2$ and $X_n(x) = \cos(nx)$ where $n = 0,1,2,3,\ldots$ Substituting $\lambda = \lambda_n = n^2$ in (9)-(10) yields

9 (11)-(12) $$T_n''(t) + 2T_n'(t) + n^2 T_n(t) = 0, \quad T_n(0) = 0 \qquad (n = 0,1,2,\ldots).$$

By the given useful facts, $T_0(t) = c_1 + c_2 e^{-2t}$, $T_1(t) = c_3 e^{-t} + c_4 t e^{-t}$, and $T_n(t) = e^{-t}\left(c_5 \cos\left(t\sqrt{n^2-1}\right) + c_6 \sin\left(t\sqrt{n^2-1}\right)\right)$ if $n \ge 2$. Applying the initial condition (12), we have, up to a constant

12 factor, that $T_0(t) = 1 - e^{-2t}$, $T_1(t) = te^{-t}$, and $T_n(t) = e^{-t}\sin\left(t\sqrt{n^2-1}\right)$ if $n \ge 2$. By

the superposition principle, a formal solution to (1)-(2)-(3)-(4) is

14 $$u(x,t)=\sum_{n=0}^{\infty} A_n X_n(x)T_n(t) = A_0(1-e^{-2t}) + A_1\cos(x)te^{-t} + \sum_{n=2}^{\infty} A_n\cos(nx)e^{-t}\sin(t\sqrt{n^2-1})$$

where $A_0, A_1, A_2, \dots$ are arbitrary constants. Then differentiating the formal solution term-by-term leads to

16 $$u_t(x,t) = 2A_0e^{-2t} + A_1\cos(x)(1-t)e^{-t} + \sum_{n=2}^{\infty} A_n\cos(nx)e^{-t}\left[\sqrt{n^2-1}\cos(t\sqrt{n^2-1}) - \sin(t\sqrt{n^2-1})\right].$$

To satisfy the nonhomogeneous boundary condition (5), we must have

17 $$\frac{1}{2}+\frac{1}{2}\cos(2x) \overset{(5)}{=} u_t(x,0) = 2A_0 + A_1\cos(x) + \sum_{n=2}^{\infty} A_n\sqrt{n^2-1}\cos(nx) \qquad \text{if } 0\le x\le\pi.$$

20 By inspection, $2A_0=\frac{1}{2}$, $\sqrt{3}A_2=\frac{1}{2}$, and all other $A_n=0$. Thus a solution to

22 (1)-(2)-(3)-(4)-(5) is $\boxed{u(x,t) = \frac{1}{4}(1-e^{-2t}) + \frac{1}{2\sqrt{3}}\cos(2x)e^{-t}\sin(t\sqrt{3})}$.

(6) (b) Let $u=u(x,t)$ satisfy (1)-(2)-(3) and consider $E(t)=\int_0^\pi\left[\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right]dx$. Then

$$\frac{dE}{dt} = \int_0^\pi \frac{\partial}{\partial t}\left[\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right]dx = \int_0^\pi\Big[u_t(x,t)u_{tt}(x,t)+\underbrace{u_x(x,t)}_{v}\overbrace{u_{xt}(x,t)}^{dv}\Big]dx =$$

$$\int_0^\pi u_t(x,t)u_{tt}\,dx + \underbrace{\cancel{u_x(x,t)u_t(x,t)\Big|_{x=0}^{\pi}}}_{0 \text{ by } (2)\text{-}(3)} - \int_0^\pi u_t(x,t)u_{xx}(x,t)\,dx = \int_0^\pi u_t(x,t)\Big[\underbrace{u_{tt}(x,t)-u_{xx}(x,t)}_{-2u_t(x,t) \text{ by } (1)}\Big]dx$$

$= -2\int_0^\pi u_t^2(x,t)dx \le 0$. Therefore $E=E(t)$ is a decreasing function for $t\ge 0$.

(6) (c) Suppose $u=v(x,t)$ were another solution of (1)-(2)-(3)-(4)-(5) and let $w(x,t)=u(x,t)-v(x,t)$ where $u=u(x,t)$ is the solution of part (a). Then w is continuous and solves (1)-(2)-(3)-(4) and (5′): $w_t(x,0)=0$ if $0\le x\le\pi$. By part (b), the energy function $E(t)=\int_0^\pi\left[\frac{1}{2}w_t^2(x,t)+\frac{1}{2}w_x^2(x,t)\right]dx$ of w is decreasing on $0\le t<\infty$, so for $t\ge 0$ $0\le E(t)\le E(0)=\int_0^\pi\left[\frac{1}{2}w_t^2(x,0)+\frac{1}{2}w_x^2(x,0)\right]dx=0$ by (4) and (5′). The vanishing theorem then implies $w_t(x,t)=0=w_x(x,t)$ if $0\le x\le\pi$, $0\le t<\infty$, and hence $w(x,t)=$ constant. But (4) implies the constant is zero and consequently $\boxed{v(x,t)=u(x,t) \text{ in the strip } 0\le x\le\pi,\ 0\le t<\infty.}$

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2 \sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a - i\xi)c} - e^{(a - i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{i(\xi - a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Math 5325

Exam III

Summer 2015

mean : 51.8

median : 51

standard deviation : 15.6

number of exams : 15

Distribution of Scores

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
87–100	A	A	1
73–86	B	B	0
60–72	C	B	2
50–59	C	C	5
0–49	F	D	7