1. (33 pts.) Use Fourier transform techniques to derive a formula for the solution to the initial value problem:

\[ u_t - u_{xx} + 2u = f(x,t) \text{ if } -\infty < x < \infty, \quad 0 < t < \infty, \]
\[ u(x,0) = 0 \text{ if } -\infty < x < \infty. \]

Let \( u = u(x,t) \) be a solution to (1) - (2) and take the Fourier transform of (1) with respect to \( x \), holding \( t \) fixed:

\[ \mathcal{F}(u_t - u_{xx} + 2u)(\xi) = \mathcal{F}(f)(\xi). \]

Then the properties of Fourier transforms gives

\[ \mathcal{F}(u_t)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + 2t\mathcal{F}(u)(\xi) = \hat{f}(\xi, t). \]

Interchanging integration with respect to \( x \) (in the Fourier transform) and differentiation with respect to \( t \) gives

\[ \frac{\partial}{\partial t} \mathcal{F}(u)(\xi) + (\xi^2 + 2t)\mathcal{F}(u)(\xi) = \hat{f}(\xi, t). \]

This is a first order, linear differential equation in the independent variable \( t \) with parameter \( \xi \). An integrating factor is

\[ \mu(t) = e \int_{0}^{t} dt \int (\xi^2 + 2t) dt = e^{t^2 + \frac{3}{2}t^2 + t}. \]

Multiplying (2) by the integrating factor and combining terms gives

\[ \frac{d}{dt} \left( e^{t^2 + \frac{3}{2}t^2} \mathcal{F}(u)(\xi) \right) = e^{t^2 + \frac{3}{2}t^2} \frac{\partial}{\partial t} \mathcal{F}(u)(\xi) + (2t + \frac{3}{2}) e^{t^2 + \frac{3}{2}t^2} \mathcal{F}(u)(\xi) = e^{t^2 + \frac{3}{2}t^2} \hat{f}(\xi, t). \]

Integrating yields

\[ e^{t^2 + \frac{3}{2}t^2} \mathcal{F}(u)(\xi) = \int_{0}^{t} e^{t^2 + \frac{3}{2}t^2} \hat{f}(\xi, \tau) d\tau + c(\xi). \]

Applying the initial condition (2) to the previous equation leads to

\[ 0 = e^{t^2 + \frac{3}{2}t^2} \mathcal{F}(u)(\xi) \bigg|_{t=0} = \left( \int_{0}^{t} e^{t^2 + \frac{3}{2}t^2} \hat{f}(\xi, \tau) d\tau + c(\xi) \right) \bigg|_{t=0} = c(\xi). \]

Therefore

\[ \mathcal{F}(u)(\xi) = e^{-\frac{t^2 + \frac{3}{2}t^2}{2}} \int_{0}^{t} e^{\frac{3}{2}t^2} \hat{f}(\xi, \tau) d\tau = \int_{0}^{t} e^{\frac{3}{2}t^2 - \frac{3}{2}(t-\tau)} \hat{f}(\xi, \tau) d\tau. \]
Apply entry I in the Fourier transform table, \( \mathcal{F}(e^{-a(t^2)}) (\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}} \), with \( t - r = \frac{1}{4a} \), or equivalently \( a = \frac{1}{4(t-r)} \), to give

\[
\mathcal{F}\left( \frac{1}{\sqrt{2(t-r)}} e^{-\frac{c^2}{4(t-r)}} \right) (\xi) = e^{-\frac{\xi^2}{2(t-r)}}.
\]

Substituting this in (**) produces

\[
\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t e^{-\frac{t^2 - \tau^2}{2}} \mathcal{F}\left( \frac{1}{\sqrt{2(t-r)}} e^{-\frac{c^2}{4(t-r)}} \right) (\xi) \mathcal{F}(f(\cdot,\tau))(\xi) d\tau.
\]

The convolution property of Fourier transforms, \( \mathcal{F}(f) \mathcal{F}(g)(\xi) = \frac{1}{\sqrt{2\pi}} \mathcal{F}(f*g)(\xi) \) then implies

\[
\mathcal{F}(u(\cdot,t))(\xi) = \int_0^t e^{-\frac{t^2 - \tau^2}{2}} \mathcal{F}\left( \frac{1}{\sqrt{4\pi(t-r)}} e^{-\frac{c^2}{4(t-r)}} * f(\cdot,\tau) \right)(\xi) d\tau
\]

\[
= \int_0^t \mathcal{F}\left( \frac{\tau^2 - \tau^2}{\sqrt{4\pi(t-r)}} e^{-\frac{c^2}{4(t-r)}} * f(\cdot,\tau) \right)(\xi) d\tau.
\]

Interchanging the order of integrations (recall that the Fourier transform is an integral over \(-\infty < x < \infty\)) yields

\[
\mathcal{F}(u(\cdot,t))(\xi) = \mathcal{F}\left( \int_0^t e^{\frac{\tau^2 - \tau^2}{2}} \frac{\tau^2 - \tau^2}{\sqrt{4\pi(t-r)}} e^{-\frac{c^2}{4(t-r)}} * f(\cdot,\tau) d\tau \right)(\xi).
\]

Apply the uniqueness theorem for Fourier transforms to obtain

\[
u(x,t) = \int_0^t \left( \frac{e^{-\frac{t^2 - \tau^2}{2}}}{\sqrt{4\pi(t-r)}} e^{-\frac{c^2}{4(t-r)}} * f(\cdot,\tau) \right)(x) d\tau.
\]

Writing the convolution product as an integral, \((g*h)(x) = \int_{-\infty}^{\infty} g(x-y)h(y)dy\), we have

\[
u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4(t-r)}}}{\sqrt{4\pi(t-r)}} f(y,\tau) dy d\tau.
\]

Equivalently, interchanging the order of integrations, we may express the solution as

\[
u(x,t) = \int_{-\infty}^{\infty} \int_0^t \frac{e^{-\frac{(x-y)^2}{4(t-r)}}}{\sqrt{4\pi(t-r)}} e^{\frac{t^2 - \tau^2}{2}} f(y,\tau) d\tau dy.
\]
2. (33 pts.) Consider the eigenvalue problem
\[ X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(1) - X(1). \]
(a) Find an equation that characterizes the positive eigenvalues of (*)
(b) Show graphically from the equation in part (a) that there is an infinite sequence of positive eigenvalues for (*).
(c) Approximate to eight decimal places the first three positive eigenvalues of (*).
(d) Is zero an eigenvalue of (*)? Support your answer.
(e) Are there any negative eigenvalues of (*)? Support your answer.
(f) Write the eigenfunction(s) corresponding to each eigenvalue of (*).

\[ (a) \text{ Suppose } \lambda > 0 \text{ in (x), say } \lambda = \beta^2 \text{ where } \beta > 0. \text{ Then (1) becomes } X''(x) + 2\beta X(x) = 0. \]

The general solution of this DE is \[ X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x). \]
Then \[ X'(x) = -\beta c_1 \sin(\beta x) + \beta c_2 \cos(\beta x). \]
So applying the BCs gives
\[ 0 = X'(0) = -\beta c_1 \cdot 0 + \beta c_2 \cdot 1 \Rightarrow c_2 = 0; \]
\[ 0 = X'(1) - X(1) = -\beta c_1 \sin(\beta) + \beta c_2 \cos(\beta) - (c_1 \cos(\beta) + c_2 \sin(\beta)) = -c_1 (\beta \sin(\beta) + \cos(\beta)). \]
Therefore, in order to have a nontrivial solution to (1)-2-3 when \[ \lambda = \beta^2 > 0, \] we must have \[ c_1 \neq 0 \text{ and } \beta \sin(\beta) + \cos(\beta) = 0, \]
or equivalently, \[ \tan(\beta) = -\frac{1}{\beta}. \]

\[ (b) \text{ Graphing the two curves } y = \tan(\beta) \text{ and } y = -\frac{1}{\beta} \text{ for } \beta > 0, \text{ we have:} \]

Clearly, there are an infinite number of intersection points of \[ y = -\frac{1}{\beta} \] with the branches of \[ y = \tan(\beta) \text{ when } \beta > 0. \]

Therefore there is an infinite sequence of positive eigenvalues \[ \lambda_n = \beta_n^2 \quad (n = 1, 2, 3, \ldots) \]
\[ \text{satisfying } \lim_{n \to \infty} \left( \lambda_n - (n\pi)^2 \right) = 0. \]

\[ (c) \text{ In order to approximate the first three positive solutions to } 0 = f(x) = \tan(x) + \frac{1}{x}, \]

we used "solve equation" in the NUMSLV menu on the HP 49G calculator with
the indicated seeds.

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</table>

(d) When $\lambda = 0$ in (1), the DE is $X''(x) = 0$ with general solution $X(x) = c_1 x + c_2$. Then $X'(x) = c_1$, so applying the BCs gives

\[
\begin{align*}
0 &= X'(0) = c_1, \\
0 &= X(1) - X(1) = c_1 - (c_1 + c_2) = -c_2.
\end{align*}
\]

Therefore $X(x) \equiv 0$ so there are no nontrivial solutions to (1)-(2)-(3) when $\lambda = 0$. That is, zero is not an eigenvalue of (1)-(2)-(3).

(e) Suppose $\lambda < 0$ in (1), say $\lambda = -\gamma^2$ where $\gamma > 0$. Then (1) becomes $X''(x) - \gamma^2 X(x) = 0$.

The general solution of this DE is $X(x) = c_1 \cosh(\gamma x) + c_2 \sinh(\gamma x)$. Then $X'(x) = \gamma c_1 \sinh(\gamma x) + \gamma c_2 \cosh(\gamma x)$ so applying the BCs gives

\[
\begin{align*}
0 &= X'(0) = \gamma c_1 \cdot 0 + \gamma c_2 \cdot 1 = \gamma c_2 \\ 0 &= X(1) - X(1) = \gamma c_1 \sinh(\gamma) + \gamma c_2 \cosh(\gamma) - (c_1 \cosh(\gamma) + c_2 \sinh(\gamma)) = c_1 (\gamma \sinh(\gamma) - \cosh(\gamma)).
\end{align*}
\]

Therefore, in order to have a negative eigenvalue we must have $c_1 \neq 0$ and $\gamma \sinh(\gamma) = \cosh(\gamma)$ i.e. $\tanh(\gamma) = \frac{1}{\gamma}$. From the graphs, it is apparent there is one intersection point of $y = \tanh(\gamma)$ and $y = \frac{1}{\gamma}$ if $\gamma > 0$.

Hence there is exactly one negative eigenvalue $\lambda_o = -\gamma_o^2$ of (1)-(2)-(3).

(f) The eigenfunctions of (1)-(2)-(3) are:

\[
\begin{align*}
X_n(x) &= \cos(\beta_n x) \quad \text{for} \quad \lambda_n = \beta_n^2 \quad (n=1,3,3,...) \\
X_0(x) &= \cosh(\gamma_0 x) \quad \text{for} \quad \lambda_o = -\gamma_0^2.
\end{align*}
\]
3. (34 pts.) (a) Find a solution to the damped wave equation

\[ u_{tt} - u_{xx} + 2u_t = 0 \quad \text{in} \quad 0 < x < \pi, \quad 0 < t < \infty, \]

satisfying

(2)-(3) \quad u_x(0, t) = 0 = u_x(\pi, t) \quad \text{for} \quad t \geq 0,

(4) \quad u(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq \pi,

and

(5) \quad u_t(x, 0) = \frac{1}{2} + \frac{1}{2} \cos(2x) \quad \text{for} \quad 0 \leq x \leq \pi.

(You may find useful the facts that \( y'' + 2y' + n^2y = 0 \) has general solution \( y_0(t) = c_1 + c_2e^{2t} \) if \( n = 0 \), \( y_1(t) = c_1e^{-t} + c_2te^{-t} \) if \( n = 1 \), and \( y_n(t) = e^{-t}\left(c_1\cos(t\sqrt{n^2-1}) + c_2\sin(t\sqrt{n^2-1})\right) \) if \( n = 2, 3, 4, \ldots \))

(b) Show that the energy \( E(t) = \int_0^\pi \left[ \frac{1}{2}u_t^2(x, t) + \frac{1}{2}u_x^2(x, t) \right] dx \) of a solution to (1)-(2)-(3) is a decreasing function for \( t \geq 0 \).

(c) Show that the solution to (1)-(2)-(3)-(4)-(5) is unique.

22. (a) We use separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem, i.e. (1)-(2)-(3)-(4), of the form \( u(x, t) = \Phi(x)T(t) \). Substituting in (1) yields

\[ \Phi''(x)T(t) - \Phi'(x)T'(t) + 2\Phi(x)T(t) = 0 \]

So

\[ -\frac{\Phi''(x)}{\Phi(x)} = -\frac{T''(t) + 2T'(t)}{T(t)} = \text{constant} = \lambda. \]  

Substituting \( u(x, t) = \Phi(x)T(t) \) in (2), (3), and (4) gives \( \Phi'(x)T(t) = 0 = \Phi(x)T'(t) \) for \( t \geq 0 \) and \( \Phi(x)T(0) = 0 \) for \( 0 \leq x \leq \pi \). Thus nontriviality of \( u \) implies

(5) \quad \Phi''(x) + \lambda \Phi(x) = 0, \quad \Phi'(x) = 0 = \Phi(x),

(6) \quad T''(t) + 2T'(t) + \lambda T(t) = 0, \quad T(0) = 0.

By Section 4.2, the eigenvalues and eigenfunctions of the Neumann problem (5)-(7)-(8) are \( \lambda_n = n^2 \) and \( \Phi_n(x) = \cos(nx) \) where \( n = 0, 1, 2, 3, \ldots \). Substituting \( \lambda = \lambda_n = n^2 \) in (9)-(10) yields

(11) \quad T_n''(t) + 2T_n'(t) + n^2T_n(t) = 0, \quad T_n(0) = 0 \quad (n = 0, 1, 2, \ldots).

By the given useful facts, \( T_0(t) = c_1 + c_2e^{-2t} \), \( T_1(t) = c_3te^{-t} + c_4e^{-t} \), and \( T_n(t) = e^{t}\left( c_5\cos(t\sqrt{n^2-1}) + c_6\sin(t\sqrt{n^2-1}) \right) \) if \( n \geq 2 \). Applying the initial condition (12), we have, up to a constant factor, that \( T_0(t) = 1 - e^{-2t} \), \( T_1(t) = te^{-t} \), and \( T_n(t) = e^{t}\sin(t\sqrt{n^2-1}) \) if \( n \geq 2 \). By
the superposition principle, a formal solution to (1)-(2)-(3)-(4) is
\[ u(x,t) = \sum_{n=0}^{\infty} A_n \cos(nx) e^{-\frac{2t}{n^2 + 1}} + \sum_{n=1}^{\infty} A_n \sin(nx) e^{-\frac{2t}{n^2 - 1}} \]
where \(A_0, A_1, A_2, \ldots\) are arbitrary constants. Then differentiating the formal solution term-by-term leads to
\[ u_t(x,t) = 2A_0 e^{-\frac{2t}{1}} + A_1 \cos(nx) e^{-\frac{2t}{n^2 + 1}} + \sum_{n=1}^{\infty} A_n \sin(nx) e^{-\frac{2t}{n^2 - 1}} \left( \sqrt{n^2 - 1} \cos(t\sqrt{n^2 - 1}) - \sin(t\sqrt{n^2 - 1}) \right). \]
To satisfy the nonhomogeneous boundary condition (5), we must have
\[ \frac{1}{2} + \cos(2x) = u_t(x,0) = 2A_0 + A_1 \cos(nx) + \sum_{n=1}^{\infty} A_n \sqrt{n^2 - 1} \cos(nx) \quad \text{if} \quad 0 \leq x \leq \pi. \]
By inspection, \(2A_0 = \frac{1}{2}, \sqrt{3}A_2 = \frac{1}{2},\) and all other \(A_n = 0.\) Thus a solution to (1)-(2)-(3)-(4)-(5) is
\[ u(x,t) = \frac{1}{4}(1 - e^{-2t}) + \frac{1}{2\sqrt{3}} \cos(2x) e^{-t}. \]
(b) Let \(u = u(x,t)\) satisfy (1)-(2)-(3) and consider \(E(t) = \int_0^\pi \left[ \frac{1}{2}u_t^2(x,t) + \frac{1}{2}u_x^2(x,t) \right] dx.\) Then
\[ \frac{dE}{dt} = \int_0^\pi \frac{\partial}{\partial t} \left[ \frac{1}{2}u_t^2(x,t) + \frac{1}{2}u_x^2(x,t) \right] dx = \int_0^\pi \left[ u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xx}(x,t) \right] dx = \int_0^\pi \left[ u_t(x,t)u_{tt}(x,t) - u_x(x,t)u_{xx}(x,t) \right] dx = -2\int_0^\pi u_t^2(x,t) dx \leq 0.\] Therefore \(E = E(t)\) is a decreasing function for \(t \geq 0.\)
(c) Suppose \(u = v(x,t)\) were another solution of (1)-(2)-(3)-(4)-(5) and let \(w(x,t) = w(x,t) - v(x,t)\) where \(u = u(x,t)\) is the solution of part (a). Then \(w\) is continuous and solves (1)-(2)-(3)-(4) and (5'): \(w_t(x,0) = 0\) if \(0 \leq x \leq \pi.\) By part (b), the energy function \(E(t) = \int_0^\pi \left[ \frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t) \right] dx\) of \(w\) is decreasing on \(0 \leq t < \infty,\) so for \(t \geq 0\)
\[ 0 \leq E(t) \leq E(0) = \int_0^\pi \left[ \frac{1}{2}w_t^2(x,0) + \frac{1}{2}w_x^2(x,0) \right] dx = 0 \quad \text{by (4) and (5').}\] The vanishing theorem then implies \(w_t(x,t) = 0 = w_x(x,t)\) if \(0 \leq x \leq \pi, 0 \leq t < \infty,\) and hence \(w(x,t) = \text{constant}.\) But (4) implies the constant is zero and consequently \(v(x,t) = u(x,t)\) in the strip \(0 \leq x \leq \pi, 0 \leq t < \infty,\)
A Brief Table of Fourier Transforms

\[ f(x) \quad \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx \]

A. \[
\begin{cases}
1 & \text{if } -b < x < b, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{\sqrt{2} \sin (b\xi)}{\pi \xi}
\]

B. \[
\begin{cases}
1 & \text{if } c < x < d, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi \sqrt{2\pi}}
\]

C. \[
\frac{1}{x^2 + a^2} \quad (a > 0)
\]
\[
\frac{\sqrt{\pi} e^{-a|\xi|}}{\sqrt{2} a}
\]

D. \[
\begin{cases}
x & \text{if } 0 < x \leq b, \\
2b - x & \text{if } b < x < 2b, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2 \sqrt{2\pi}}
\]

E. \[
\begin{cases}
e^{-ax} & \text{if } x > 0, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{1}{(a + i\xi) \sqrt{2\pi}}
\]

F. \[
\begin{cases}
e^{ax} & \text{if } b < x < c, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{e^{(a - i\xi)x} - e^{(a - i\xi)c}}{(a - i\xi) \sqrt{2\pi}}
\]

G. \[
\begin{cases}
e^{i\alpha x} & \text{if } -b < x < b, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{\sqrt{2} \sin (b(\xi - a))}{\pi \xi - a}
\]

H. \[
\begin{cases}
e^{i\alpha x} & \text{if } c < x < d, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
\frac{e^{i(\xi - a)x} - e^{i(\xi - a)d}}{i(\xi - a) \sqrt{2\pi}}
\]

I. \[
e^{-ax^2} \quad (a > 0)
\]
\[
\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}
\]

J. \[
\begin{cases}
\sin(ax) & \text{if } (\xi) \geq a, \\
\sqrt{\pi} & \text{if } (\xi) < a.
\end{cases}
\]
\[
\frac{1}{\sqrt{2}}
\]
Math 5325
Exam III
Summer 2015

mean: 51.8
median: 51
standard deviation: 15.6
number of exams: 15

Distribution of Scores

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