Mathematics 325
Homework Assignment 3

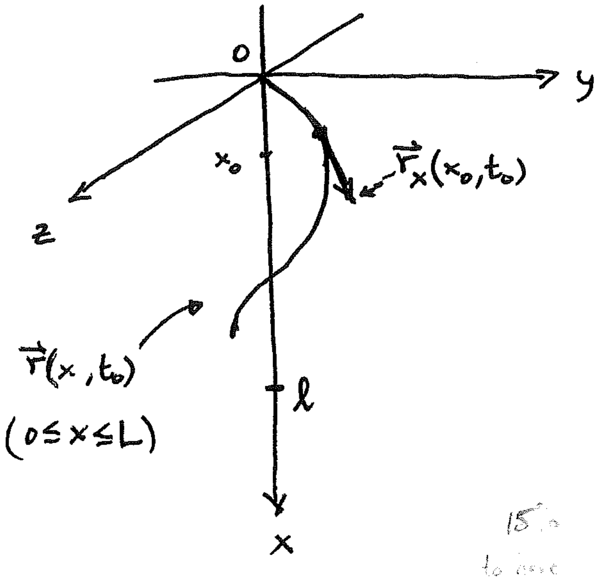
Due Date: ____________________

Name:___________________________

Work exercise 2 on page 18.

2. A flexible chain of length l is hanging from one end $x = 0$ but oscillates horizontally. Let the x – axis point downward and the u – axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the partial differential equation satisfied by the chain.

Solution using configurations, contact forces, etc. as introduced in lecture:



Let $\vec{r}(x,t)$ denote the position of particle x in the flexible chain at time t. The development of the model in class is valid here and results in the equation of motion

15% to here

$$(2) \qquad \vec{n}_x(x,t) + \vec{f}(x,t) = \rho(x)\vec{r}_{tt}(x,t)$$

where $\vec{n}(x,t)$ is the contact force on the chain at (x,t), $\rho(x)$ is mass density per unit length at x in the reference configuration, and $\vec{f}(x,t)$ is the force per unit reference length exerted by all other agents beside the contact force. We will assume that $\vec{f}(x,t) = \vec{0}$. The problem description means the contact force at (x,t) has the form

30%

$$(3) \qquad \vec{n}(x,t) = |\vec{n}(x,t)| \frac{\vec{r}_x(x,t)}{|\vec{r}_x(x,t)|}$$

where $|\vec{n}(x,t)|$, the magnitude of the contact force, is equal to the weight of the chain below x:

45%

$$(4) \qquad |\vec{n}(x,t)| = \int_x^{\ell} \rho(s) g \, ds$$

where g is the (constant) acceleration of gravity. On the other hand, although the chain is flexible, it resists stretching, so

60% to here

$$(5) \qquad |\vec{r}_x(x,t)| = \lim_{\Delta x \to 0} \left| \frac{\vec{r}(x+\Delta x,t) - \vec{r}(x,t)}{\Delta x} \right| = 1$$

for all (x,t). (Geometrically, this means that $\vec{r}_x(x,t)$ is a <u>unit</u> vector tangent to the configuration curve $\vec{r}(\cdot,t)$ at each point $x \in (0,\ell)$.) Substituting from (4) and (5) into (3) and then into (2) gives

75 %

$$\frac{\partial}{\partial x}\left\{ \left(\int_x^\ell \rho(s) g\, ds \right) \vec{r}_x(x,t) \right\} = \rho(x)\, \vec{r}_{tt}(x,t).$$

For example, writing $\vec{r}(x,t) = w(x,t)\hat{\imath} + v(x,t)\hat{\jmath} + u(x,t)\hat{k}$, we see that the equation of motion for the y-component of the chain is given by

90 %

$$\boxed{\frac{\partial}{\partial x}\left\{ \left(\int_x^\ell \rho(s) g\, ds \right) \frac{\partial v}{\partial x}(x,t) \right\} = \rho(x) \frac{\partial^2 v}{\partial t^2}(x,t).}$$

For a homogeneous chain with constant linear density $\rho(x) = \rho_0 > 0$, this equation becomes

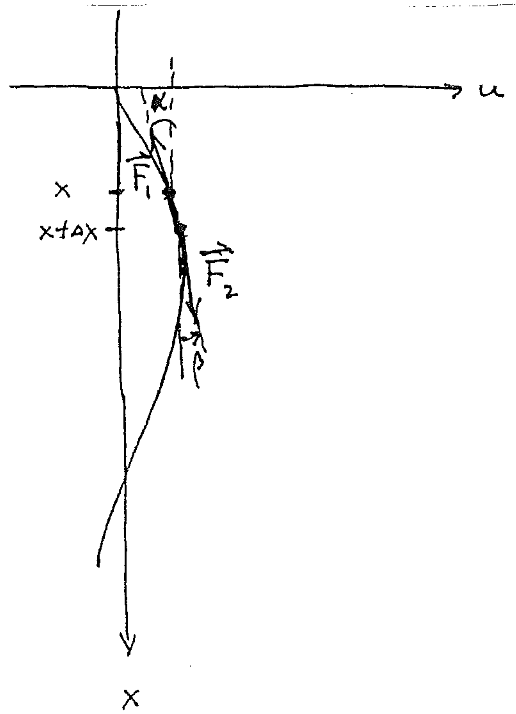
$$\frac{\partial}{\partial x}\left\{ \cancel{\rho_0} g(\ell - x) \frac{\partial v}{\partial x}(x,t) \right\} = \cancel{\rho_0} \frac{\partial^2 v}{\partial t^2}(x,t)$$

or equivalently

100 %

$$\boxed{g \frac{\partial}{\partial x}\left\{ (\ell - x) \frac{\partial v}{\partial x}(x,t) \right\} = \frac{\partial^2 v}{\partial t^2}(x,t).}$$

Solution using geometrical ideas similar to those in our textbook, section 1.3, pp. 11–12.



Fix a time $t > 0$ and consider the segment of chain between x and $x + \Delta x$. The forces $\vec{F}_1$ and $\vec{F}_2$ that act on the endpoints of this segment are tangential to the segment there. Let α and β denote the angles that $\vec{F}_1$ and $\vec{F}_2$ make with respect to vertical. Then

10% to here

$$(1) \qquad \tan(\alpha) = u_x(x,t)$$

and consequently

20% to here. (2) $$\cos(\alpha) = \frac{1}{\sqrt{1+u_x^2(x,t)}} \quad \text{and} \quad \sin(\alpha) = \frac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}.$$

Similarly,

30% (3) $$\cos(\beta) = \frac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}} \quad \text{and} \quad \sin(\beta) = \frac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}.$$

Applying Newton's second law of motion, $\vec{F}_{net} = m\vec{a}$, to the segment of chain between x and $x+\Delta x$ and resolving this into vertical and horizontal components yields

40% (4) $$|\vec{F}_2|\cos(\beta) - |\vec{F}_1|\cos(\alpha) = 0$$

(This is not a very good assumption and it should be avoided in the subsequent derivation!)

50% (5) $$|\vec{F}_2|\sin(\beta) - |\vec{F}_1|\sin(\alpha) = \int_x^{x+\Delta x} u_{tt}(\xi,t)\rho\, d\xi$$

where ρ is the (constant) linear density of the chain. By the assumption in the third sentence of the problem description,

60% (6) $$|\vec{F}_1| = \text{weight of chain below } x = (\ell - x)\rho g$$

and

70% (7) $$|\vec{F}_2| = \text{weight of chain below } x+\Delta x = (\ell-(x+\Delta x))\rho g$$

where g is the (constant) acceleration of gravity. Substituting into (5) from (2), (3), (6), and (7) gives

80% to here. (8)

$$\frac{(l-(x+\Delta x))\rho g u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}} - \frac{(l-x)\rho g u_x(x,t)}{\sqrt{1+u_x^2(x,t)}} = \int_x^{x+\Delta x} u_{tt}(\xi,t)\rho\, d\xi$$

Dividing both sides of (8) by Δx and taking a limit as $\Delta x \to 0$ yields

90% (9)

$$\rho g \frac{\partial}{\partial x}\left(\frac{(l-x)u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}\right) = u_{tt}(x,t)\rho .$$

For small oscillations, $\sqrt{1+u_x^2(x,t)}$ is approximately 1, so cancelling ρ and using this approximation in (9) yields

100%

$$\boxed{g\left[(l-x)u_x(x,t)\right]_x = u_{tt}(x,t)} .$$

(Here g is the acceleration of gravity and l is the length of the chain.)

Equivalently,

$$-g u_x(x,t) + g(l-x)u_{xx}(x,t) = u_{tt}(x,t) .$$