Work exercise 2 on page 18.
2. A flexible chain of length $l$ is hanging from one end $x = 0$ but oscillates horizontally. Let the $x$–axis point downward and the $u$–axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the partial differential equation satisfied by the chain.
Solution using configurations, contact forces, etc. as introduced in
lecture:

Let \( \mathbf{r}(x,t) \) denote the position of
particle \( x \) in the flexible chain at time \( t \).
The development of the model in class
is valid here and results in the
equation of motion

\[
(2) \quad \ddot{\mathbf{r}}_x(x,t) + \mathbf{f}(x,t) = \rho(x) \mathbf{v}^c_t(x,t)
\]

where \( \mathbf{v}(x,t) \) is the contact force on the chain at \( (x,t) \), \( \rho(x) \) is
mass density per unit length at \( x \) in the reference configuration, and
\( \mathbf{f}(x,t) \) is the force per unit reference length exerted by all other agents beside
the contact force. We will assume that \( \mathbf{f}(x,t) = \mathbf{0} \). The problem
description means the contact force at \( (x,t) \) has the form

\[
(3) \quad \mathbf{v}(x,t) = \frac{\mathbf{v}(x,t)}{|\mathbf{v}(x,t)|}
\]

where \( |\mathbf{v}(x,t)| \), the magnitude of the contact force, is equal to the
weight of the chain below \( x \):

\[
(4) \quad |\mathbf{v}(x,t)| = \int_{x}^{x} \rho(s)g \, ds
\]

where \( g \) is the (constant) acceleration of gravity. On the other hand,
although the chain is flexible, it resists stretching, so

$$|\vec{r}_x(x,t)| = \lim_{\Delta x \to 0} \left| \frac{\vec{r}(x+\Delta x,t) - \vec{r}(x,t)}{\Delta x} \right| = 1$$

for all $(x,t)$. (Geometrically, this means that $\vec{r}_x(x,t)$ is a unit vector tangent to the configuration curve $\vec{r}(\cdot,t)$ at each point $x \in (0, L)$.)

Substituting from (4) and (5) into (3) and then into (2) gives

$$\frac{\partial}{\partial x} \left\{ \left( \int_x^L \rho(s) g ds \right) \vec{r}_x(x,t) \right\} = \rho(x) \vec{r}_{tt}(x,t).$$

For example, writing $\vec{r}(x,t) = w(x,t) \hat{i} + v(x,t) \hat{j} + u(x,t) \hat{k}$, we see that the equation of motion for the $y$-component of the chain is given by

$$\frac{\partial}{\partial x} \left\{ \left( \int_x^L \rho(s) g ds \right) \frac{\partial v}{\partial x} (x,t) \right\} = \rho(x) \frac{\partial^2 v}{\partial t^2} (x,t).$$

For a homogeneous chain with constant linear density $\rho(x) = \rho_0 > 0$, this equation becomes

$$\frac{\partial}{\partial x} \left\{ \rho_0 g(l-x) \frac{\partial v}{\partial x} (x,t) \right\} = \rho_0 \frac{\partial^2 v}{\partial t^2} (x,t)$$

or equivalently

$$\rho \frac{\partial}{\partial x} \left\{ (l-x) \frac{\partial v}{\partial x} (x,t) \right\} = \frac{\partial^2 v}{\partial t^2} (x,t).$$
Solution using geometrical ideas similar to those in our textbook, Section 1.3, pp. 11-12.

Fix a time \( t > 0 \) and consider the segment of chain between \( x \) and \( x + \Delta x \). The forces \( \vec{F}_1 \) and \( \vec{F}_2 \) that act on the endpoints of this segment are tangential to the segment there. Let \( \alpha \) and \( \beta \) denote the angles that \( \vec{F}_1 \) and \( \vec{F}_2 \) make with respect to vertical. Then

\[
\tan(\alpha) = u_x(x, t) \quad (1)
\]

and consequently
(2) \[
\cos(\alpha) = \frac{1}{\sqrt{1+u_x^2(x,t)}} \quad \text{and} \quad \sin(\alpha) = \frac{u_x(x,t)}{\sqrt{1+u_x^2(x,t)}}.
\]

Similarly,

(3) \[
\cos(\beta) = \frac{1}{\sqrt{1+u_x^2(x+\Delta x,t)}} \quad \text{and} \quad \sin(\beta) = \frac{u_x(x+\Delta x,t)}{\sqrt{1+u_x^2(x+\Delta x,t)}}.
\]

Applying Newton's second law of motion, \( \vec{F}_{\text{net}} = ma \), to the segment of chain between \( x \) and \( x+\Delta x \) and resolving this into vertical and horizontal components yields

(4) \[
|\vec{F}_2| \cos(\beta) - |\vec{F}_1| \cos(\alpha) = 0 \quad x+\Delta x
\]

(5) \[
|\vec{F}_2| \sin(\beta) - |\vec{F}_1| \sin(\alpha) = \int_{x}^{x+\Delta x} u_{xx}(\xi,t) \rho \, d\xi
\]

where \( \rho \) is the (constant) linear density of the chain. By the assumption in the third sentence of the problem description,

(6) \[
|\vec{F}_1| = \text{weight of chain below } x = (l-x) \rho g
\]

and

(7) \[
|\vec{F}_2| = \text{weight of chain below } x+\Delta x = (l-(x+\Delta x)) \rho g
\]

where \( g \) is the (constant) acceleration of gravity. Substituting into (5) from (2), (3), (6), and (7) gives
\[
\frac{(l - (x + \Delta x)) \rho g u_x(x + \Delta x, t)}{\sqrt{1 + u_x^2(x + \Delta x, t)}} - \frac{(l - x) \rho g u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}} = \int_{x}^{x + \Delta x} \int_{t}^{t + \Delta t} u_{xt}(\xi, \eta) \rho d\xi d\eta
\]

Dividing both sides of (8) by \(\Delta x\) and taking a limit as \(\Delta x \to 0\) yields

\[
\rho g \frac{\partial}{\partial x} \left( \frac{(l - x) u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}} \right) = u_{tt}(x, t) \rho.
\]

For small oscillations, \(\sqrt{1 + u_x^2(x, t)}\) is approximately 1, so cancelling \(\rho\) and using this approximation in (9) yields

\[
g \left[ (l - x) u_x(x, t) \right]_x = u_{tt}(x, t).
\]

(Here \(g\) is the acceleration of gravity and \(l\) is the length of the chain.)

Equivalently,

\[-gu_x(x, t) + g(l - x)u_{xx}(x, t) = u_{tt}(x, t).\]