Fourier Transforms and Diffusions in the Upper Halfplane

<u>Def</u>: Let f be an absolutely integrable function on $(-\infty, \infty)$. (I.e. let $f \in L^1(\mathbb{R})$.)
The <u>Fourier transform</u> of f is the function given by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \qquad (-\infty < \xi < \infty).$$

<u>Ex 1</u> Compute the Fourier transform of the function

$$\chi_{(c,d)}(x) = \begin{cases} 1 & \text{if } x \in (c,d), \\ 0 & \text{o.w.} \end{cases}$$

<u>Solution</u>:

$$\hat{\chi}_{(c,d)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{(c,d)}(x) e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_c^d e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\xi x}}{-i\xi} \Bigg|_{x=c}^{x=d} \right) \qquad (\text{provided } \xi \neq 0)$$

Formula B.

$$\boxed{\widehat{\chi_{(c,d)}}(\xi) = \frac{e^{-i\xi c} - e^{-i\xi d}}{i\xi\sqrt{2\pi}} \qquad (\xi \neq 0)}$$

Notes: ① $\hat{\chi}_{(c,d)}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{(c,d)}(x) e^{-i0x}\, dx = \frac{d-c}{\sqrt{2\pi}} \overset{\text{l'Hôpital}}{=} \lim_{\xi \to 0} \widehat{\chi_{(c,d)}}(\xi)$.

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② The special case $c=-b$ and $d=b$ in <u>Ex 1</u> yields

Formula A.

$$\boxed{\widehat{\chi_{(-b,b)}}(\xi) =} \frac{e^{ib\xi} - e^{-ib\xi}}{i\xi\sqrt{2\pi}} = \frac{2i\sin(b\xi)}{i\xi\sqrt{2\pi}} = \boxed{\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}}.$$

Unnormalized sinc function $\xi \mapsto \frac{\sin(\xi)}{\xi}$

(Used in digital signal processing and information theory)

Properties of the Fourier Transform: $\overset{\mathcal{F}(f)(\xi)}{\hat{f}(\xi)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$

(Linearity) ① $\widehat{(c_1 f_1 + c_2 f_2)}(\xi) = c_1\hat{f}_1(\xi) + c_2\hat{f}_2(\xi)$ (Valid if f_1, f_2 are absolutely integrable and $c_1, c_2 \in \mathbb{R}$.)

(Transform of a derivative) ② $\widehat{f'}(\xi) = i\xi\hat{f}(\xi)$ (Valid if f is differentiable and f and f' are ~~[illegible]~~ absolutely integrable)

(Derivative of a transform) ③ $\frac{d}{d\xi}\mathcal{F}(f)(\xi) = -i\,\mathcal{F}\big(xf(x)\big)(\xi)$ (Valid if f and $x \mapsto xf(x)$ are ~~[illegible]~~ absolutely integrable.)

(Convolutions) ④ $\widehat{f*g}(\xi) = \sqrt{2\pi}\,\hat{f}(\xi)\hat{g}(\xi)$ (Valid if f, g are absolutely integrable)

(Inversion Theorem) ⑤ $f(x) = \lim\limits_{M\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-M}^{M}\hat{f}(\xi)e^{i\xi x}\,d\xi$ (Valid if f is continuous, piecewise smooth, and absolutely integrable.)

See also the shifting theorems (exercises #4, 6 for Fourier transforms).

Property ①:

$$\widehat{(c_1f_1+c_2f_2)}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[c_1f_1(x)+c_2f_2(x)\right]e^{-i\xi x}\,dx$$

$$= c_1\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)e^{-i\xi x}\,dx + c_2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(x)e^{-i\xi x}\,dx$$

$$= c_1\hat{f}_1(\xi) + c_2\hat{f}_2(\xi).$$

Property ②:

$$\widehat{f'}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\xi x}\,dx$$

$$= \lim_{\substack{M\to\infty\\ N\to-\infty}}\frac{1}{\sqrt{2\pi}}\int_{N}^{M} f'(x)e^{-i\xi x}\,dx$$

Parts: $v = e^{-i\xi x}$ $\quad dv = -i\xi e^{-i\xi x}dx$

$dV = f'(x)dx$ $\quad V = f(x)$

$$(\dagger)\qquad = \lim_{\substack{M\to\infty\\ N\to+\infty}}\left\{\frac{f(M)e^{-i\xi M}}{\sqrt{2\pi}} - \frac{f(N)e^{+i\xi N}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}}\int_{-N}^{M} i\xi f(x)e^{-i\xi x}\,dx\right\}$$

Since f' is absolutely integrable on $(-\infty,\infty)$, to each $\varepsilon > 0$ there corresponds

$M_0 = M_0(\varepsilon) > 0$ such that

$$|f(M_2) - f(M_1)| = \left| \int_{M_1}^{M_2} f'(x)\,dx \right| \le \int_{M_1}^{M_2} |f'(x)|\,dx < \varepsilon$$

for all $M_1, M_2 \ge M_0$. It follows that $\lim\limits_{M\to\infty} f(M)$ exists.

A similar argument shows that $\lim\limits_{N\to+\infty} f(-N)$ exists. However, f is absolutely integrable on $(-\infty,\infty)$ so $\lim\limits_{M\to\infty} f(M) = 0 = \lim\limits_{N\to+\infty} f(-N)$.

Returning to $(\dagger)$, we see that

$$\lim_{M\to\infty} \frac{f(M)\overbrace{e^{-i\xi M}}^{\text{bounded}}}{\sqrt{2\pi}} = 0 \quad \text{and} \quad \lim_{N\to+\infty} \frac{f(-N)\overbrace{e^{+i\xi N}}^{\text{bounded}}}{\sqrt{2\pi}} = 0.$$

Consequently, $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$.

Property 3: $\frac{d}{d\xi}\mathcal{F}(f)(\xi) = \frac{d}{d\xi}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\,dx \right)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi}\left(f(x) e^{-i\xi x} \right) dx \qquad \left(\begin{array}{r}\text{Theorem 2, p.390,}\\ \text{Appendix A.3}\end{array}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ix f(x) e^{-i\xi x}\,dx$$

$$= -i\,\mathcal{F}\big(x f(x)\big)(\xi).$$

Ex 2 Compute the Fourier transform of the function $f(x) = e^{-ax^2}$ where a is a positive constant.

Solution:

$$f(x) = e^{-ax^2}$$

$$f'(x) = -2ax\,e^{-ax^2}$$

$$\therefore \quad f'(x) + 2axf(x) = 0 \qquad (-\infty < x < \infty)$$

We take the Fourier transform of both sides of this identity and use linearity:

$$\mathcal{F}(f')(\xi) + 2a\,\mathcal{F}(xf(x))(\xi) = 0$$

Applying properties ② and ③ gives

$$i\xi\hat{f}(\xi) + 2ai\frac{d}{d\xi}\hat{f}(\xi) = 0$$

$$\Rightarrow \quad \int \frac{d\hat{f}(\xi)}{f(\xi)} = \int -\frac{\xi}{2a}\,d\xi$$

$$\ln \hat{f}(\xi) = -\frac{\xi^2}{4a} + c$$

$$\hat{f}(\xi) = A e^{-\frac{\xi^2}{4a}} \qquad (A = e^c)$$

To find A, we evaluate at $\xi = 0$:

$$A = \hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i0x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\,dx \qquad \left\{\begin{array}{l}\text{Let } p = x\sqrt{a}\\ dp = \sqrt{a}\,dx\end{array}\right.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-p^2}\frac{dp}{\sqrt{a}} = \frac{1}{\sqrt{2\pi a}}\sqrt{\pi} = \frac{1}{\sqrt{2a}}$$

$$\therefore \quad \boxed{\mathcal{F}\left(e^{-ax^2}\right)(\xi) = \frac{e^{-\frac{\xi^2}{4a}}}{\sqrt{2a}}} \qquad \text{Formula I.}$$

Note: $\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)(\xi) = e^{-\frac{\xi^2}{2}}$. Therefore $f(x) = e^{-\frac{x^2}{2}}$ is a fixed point of the Fourier transform operator; i.e. $\mathcal{F}f = f$.

Take $a = \frac{1}{2}$ in Formula I to get

Convolutions

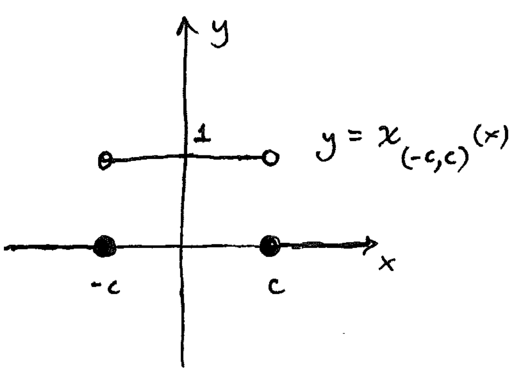
Def: Let f and g be absolutely integrable on $(-\infty, \infty)$. Their <u>convolution product</u> (denoted $f*g$) is the function given by

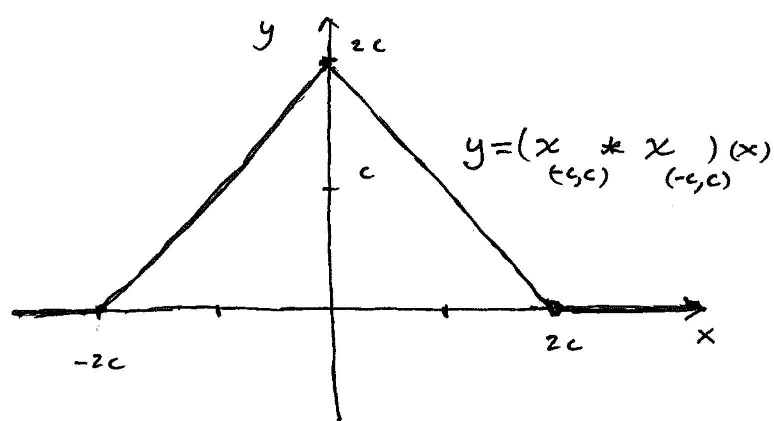
$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \qquad (-\infty < x < \infty).$$

Ex 3 Compute the convolution product of $\chi_{(-c,c)}$ with itself.

$$\begin{cases} -c < x-y < c \\ \Updownarrow \\ -c < y-x < c \\ \Updownarrow \\ x-c < y < x+c \end{cases}$$

Soln:
$$\begin{aligned}
(\chi_{(-c,c)} * \chi_{(-c,c)})(x) &= \int_{-\infty}^{\infty} \chi_{(-c,c)}(x-y)\,\chi_{(-c,c)}(y)\,dy \\
&= \int_{-\infty}^{\infty} \chi_{(x-c,x+c)}(y)\,\chi_{(-c,c)}(y)\,dy \\
&= \int_{-\infty}^{\infty} \chi_{(x-c,x+c)\cap(-c,c)}(y)\,dy \\
&= \text{length of } \{(x-c,x+c)\cap(-c,c)\} \\
&= \begin{cases} 2c - |x| & \text{if } |x| \le 2c, \\ 0 & \text{o.w.} \end{cases}
\end{aligned}$$





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Note: Ex 3 illustrates the basic principle that convolution is a "smoothing" operation. Moreover $f*g$ inherits the smoothness properties of either factor.

E.g. if f is differentiable (with f, f', and g ~~are~~ absolutely integrable) then so $f*g$ is differentiable with $(f*g)'(x) = (f'*g)(x)$.

Elementary Properties of the Convolution Product

1. $f*g = g*f$

2. $(c_1f_1+c_2f_2)*g = c_1(f_1*g) + c_2(f_2*g)$

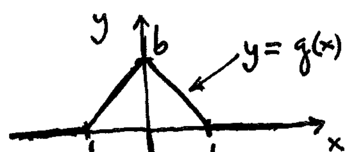
3. If f and g are absolutely integrable then so is $f*g$:
$$\int_{-\infty}^{\infty} |(f*g)(x)|\,dx \le \int_{-\infty}^{\infty}|f(x)|\,dx \cdot \int_{-\infty}^{\infty}|g(y)|\,dy.$$

Property ④ of Fourier Transforms: $\widehat{f*g}(\xi) = \sqrt{2\pi}\,\hat{f}(\xi)\hat{g}(\xi)$

④:
$$\widehat{f*g}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)e^{-i\xi x}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right)e^{-i\xi x}\,dx$$

Let $z = x-y$. Then $dz = dx$.

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)e^{-i\xi(x-y)}\,dx\right)g(y)e^{-i\xi y}\,dy$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(z)e^{-i\xi z}\,dz\right)g(y)e^{-i\xi y}\,dy$$
$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} f(z)e^{-i\xi z}\,dz\right)\int_{-\infty}^{\infty} g(y)e^{-i\xi y}\,dy$$
$$= \sqrt{2\pi}\,\hat{f}(\xi)\hat{g}(\xi).$$



Ex 4] Compute the Fourier transform of $g(x) = \begin{cases} b-|x| & \text{if } |x|\le b, \\ 0 & \text{if } |x|>b. \end{cases}$

Solution: Recall from Ex 3 with $c=\frac{b}{2}$ that $g(x) = \chi_{(-\frac{b}{2},\frac{b}{2})} * \chi_{(-\frac{b}{2},\frac{b}{2})}(x)$.

Verification of Property 3 of Convolutions:

$$\int_{-\infty}^{\infty} |(f * g)(x)|\, dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \right| dx$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-y)g(y)|\, dy\, dx$$

$$= \int_{-\infty}^{\infty} |g(y)| \left(\int_{-\infty}^{\infty} |f(x-y)|\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} |g(y)| \left[\int_{-\infty}^{\infty} |f(z)|\, dz \right] dy$$

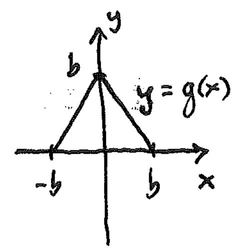
$$= \left(\int_{-\infty}^{\infty} |f(z)|\, dz \right) \int_{-\infty}^{\infty} |g(y)|\, dy$$

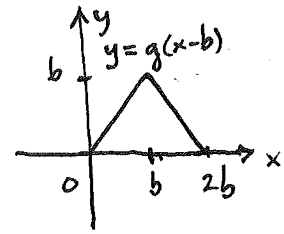
Therefore $\hat{g}(\xi) = \widehat{\chi_{(-\frac{b}{2},\frac{b}{2})} * \chi_{(-\frac{b}{2},\frac{b}{2})}}(\xi)$

$$= \sqrt{2\pi}\, \hat{\chi}_{(-\frac{b}{2},\frac{b}{2})}(\xi)\, \hat{\chi}_{(-\frac{b}{2},\frac{b}{2})}(\xi)$$

$$= \sqrt{2\pi}\left(\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi/2)}{\xi}\right)\left(\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi/2)}{\xi}\right) \qquad (\text{by } \underline{\text{Ex 1}})$$

$$= \boxed{2\sqrt{\frac{2}{\pi}}\frac{\sin^2(b\xi/2)}{\xi^2}}$$





Note: $g(x-b) = \begin{cases} x & \text{if } 0 < x \leq b, \\ 2b-x & \text{if } b < x < 2b, \\ 0 & \text{o.w.}, \end{cases}$

so, by Ex 4 and shifting on the x-axis (exercise #4),

$$\mathcal{F}\{g(x-b)\}(\xi) = e^{-i\xi b}\mathcal{F}(g)(\xi)$$

$$= 2\sqrt{\frac{2}{\pi}}\, e^{-i\xi b}\left(\frac{e^{\frac{ib\xi}{2}} - e^{-\frac{ib\xi}{2}}}{2i}\right)^2 \cdot \frac{1}{\xi^2}$$

$$= 2\sqrt{\frac{2}{\pi}}\, e^{-i\xi b}\left(\frac{e^{ib\xi} - 2 + e^{-ib\xi}}{-4}\right)\cdot \frac{1}{\xi^2}$$

$$= \frac{-1 + 2e^{-i\xi b} - e^{-2i\xi b}}{\xi^2\sqrt{2\pi}} \qquad \text{Formula D.}$$

omit if pressed for time.

Property (5): For a proof of the inversion theorem see: R. Goldberg, Fourier Transforms, Cambridge U. Press, 1962, pp. 10-13.

or M. Pinsky, PDE's and BVP's with Applications (2nd ed.), McGraw-Hill, 1991, pp. 253-4.

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Solution of the Diffusion Equation in the Upper Halfplane
(via Fourier transforms)

Consider the I.V.P.

$$(*)\quad \begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = \varphi(x) & \text{for } -\infty < x < \infty. \end{cases}$$

We will use Fourier transform techniques to derive the solution

$$(**)\qquad u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)}{4kt}}}{\sqrt{4k\pi t}}\varphi(y)\,dy + \int_0^t\int_0^\infty \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y,\tau)\,dy\,d\tau .$$

The derivation will be "formal", i.e. we won't worry about the validity of interchanging integration and differentiation, convergence of improper integrals, etc.

STEP 1. Convert the PDE to an ODE problem by taking the Fourier transform.

We begin by taking the Fourier transform of the p.d.e. in $(*)$ with respect to the variable x.

$$\mathcal{F}\big(u_t(\cdot,t) - ku_{xx}(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi) = \hat{f}(\xi,t)$$

Using properties ① and ② of the Fourier transform,

$$\mathcal{F}(u_t)(\xi) - k\mathcal{F}(u_{xx})(\xi) = \hat{f}(\xi,t)$$

$$\mathcal{F}(u_t)(\xi) - k(i\xi)^2\mathcal{F}(u)(\xi) = \hat{f}(\xi,t)$$

But $\mathcal{F}(u_t)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)e^{-i\xi x}\,dx = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx\right) = \frac{\partial}{\partial t}\mathcal{F}(u)(\xi)$

so

$$(\dagger)\qquad \frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2\mathcal{F}(u)(\xi) = \hat{f}(\xi,t).$$

(Observe that via the Fourier transform, we have converted the p.d.e. in (*) in the variables x and t into an o.d.e. in the variable t, with parameter ξ.)

STEP 2. Solve the ODE problem

An integrating factor for the first-order linear equation (†) is

$$e^{\int k\xi^2 dt} = e^{k\xi^2 t + \cancel{c}^{\,0}}.$$

Multiplying (†) by this integrating factor yields

$$e^{k\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}(u)(\xi) = e^{k\xi^2 t}\hat{f}(\xi,t)$$

$$\frac{\partial}{\partial t}\left(e^{k\xi^2 t}\mathcal{F}(u)(\xi)\right) = e^{k\xi^2 t}\hat{f}(\xi,t).$$

Consequently, $$e^{k\xi^2 t}\mathcal{F}(u)(\xi) = \int_0^t \hat{f}(\xi,\tau)e^{k\xi^2\tau}\,d\tau + A(\xi)$$ or

$$\mathcal{F}(u)(\xi) = \int_0^t \hat{f}(\xi,\tau)e^{-k\xi^2(t-\tau)}\,d\tau + A(\xi)e^{-k\xi^2 t}$$

In order to determine the factor $A(\xi)$ we use the initial condition in (*):

$$A(\xi) = \left[A(\xi)e^{-k\xi^2 t} + \int_0^t \hat{f}(\xi,\tau)e^{-k\xi^2(t-\tau)}\,d\tau\right]\Bigg|_{t=0} = \mathcal{F}(u(\cdot,t))(\xi)\Bigg|_{t=0} = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(\varphi)(\xi).$$

Therefore

$$(\dagger\dagger)\qquad \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)e^{-k\xi^2 t} + \int_0^t \hat{f}(\xi,\tau)e^{-k\xi^2(t-\tau)}\,d\tau \equiv U_1(\xi,t)+U_2(\xi,t)$$

STEP 3. Convert the solution of the ODE problem into a solution of the PDE.
(I.e. compute the inverse Fourier transform of the ODE's solution.)

Using formula I in the table of Fourier transforms,

$$\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{e^{-\xi^2/4a}}{\sqrt{2a}},$$

with $a = \frac{1}{4kt}$ yields

$$e^{-kt\xi^2} = \mathcal{F}\left(\sqrt{\frac{1}{2kt}}\, e^{\frac{-(\cdot)^2}{4kt}} \right)(\xi).$$

Thus $U_1(\xi,t) = \mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\frac{1}{\sqrt{2kt}} e^{\frac{-(\cdot)^2}{4kt}} \right)(\xi)$. Using property ④ gives

$$(\dagger\dagger\dagger) \qquad U_1(\xi,t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}\left(\varphi * \frac{1}{\sqrt{2kt}} e^{\frac{-(\cdot)^2}{4kt}} \right)(\xi)$$

$$= \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} \right)(\xi).$$

Using formula I in the table of Fourier transforms with $a = \frac{1}{4k(t-\tau)}$ gives

$$e^{-k\xi^2(t-\tau)} = \mathcal{F}\left(\sqrt{\frac{1}{2k(t-\tau)}}\, e^{-\frac{(\cdot)^2}{4k(t-\tau)}} \right)(\xi).$$

Therefore

$$U_2(\xi,t) = \int_0^t \hat{f}(\xi,\tau) e^{-k\xi^2(t-\tau)}\, d\tau$$

$$= \int_0^t \mathcal{F}(f(\cdot,\tau))(\xi)\, \mathcal{F}\left(\frac{1}{\sqrt{2k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} \right)(\xi)\, d\tau.$$

Property ④ and interchanging the order of integration then yields

$$(\dagger\dagger\dagger\dagger) \qquad U_2(\xi,t) = \int_0^t \frac{1}{\sqrt{2\pi}} \mathcal{F}\left(f(\cdot,\tau) * \frac{1}{\sqrt{2k(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} \right)(\xi)\, d\tau$$

$$= \mathcal{F}\left(\int_0^t f(\cdot,\tau) * \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}}\, d\tau \right)(\xi).$$

Consequently, substituting from $(\dagger\dagger\dagger)$ and $(\dagger\dagger\dagger\dagger)$ into $(\dagger\dagger)$ leads to

$$\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi) + \mathcal{F}\left(\int_0^t f(\cdot,\tau) * \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} d\tau\right)(\xi)$$

$$= \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} + \int_0^t f(\cdot,\tau) * \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} d\tau\right)(\xi)$$

The inversion theorem – i.e. property ⑤ – implies two continuous functions whose Fourier transforms agree for every $\xi \in \mathbb{R}$ are actually identical functions of $x \in \mathbb{R}$. (This result is known as the uniqueness theorem for Fourier transforms.) Therefore

$$u(x,t) = \left(\varphi * \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(\cdot)^2}{4kt}} + \int_0^t f(\cdot,\tau) * \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(\cdot)^2}{4k(t-\tau)}} d\tau\right)(x)$$

for all $x \in \mathbb{R}$ (and $t>0$). Writing out the convolutions in terms of integrals gives (**):

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y,\tau)\,dy\,d\tau .}$$