Sec. 1.1  What is a PDE?

(Use a rubber band prop.) An elastic string is stretched to a length L and fixed at its endpoints. The string is distorted and then released at a certain instant, say $t = 0$. We seek the transverse displacement $u(x,t)$ of the string at position $x$ in $[0,L]$ and time $t \geq 0$.

We will see in Sec. 1.3 that for "small" displacements, the equation governing the motion is (approximately)

\begin{equation}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (1-D \text{ Wave Equation})
\end{equation}

where $c$ is a positive constant that depends on the physical properties (tension and density) of the string.

[Need to informally define "PDE" and "solution" of a PDE here.]

(over for formal)
Ex. 1 (a) Verify that for each \( n = 1, 2, 3, \ldots \) the function
\[
 u_n(x, t) = \cos \left( \frac{n \pi c t}{L} \right) \sin \left( \frac{n \pi x}{L} \right)
\]
is a solution of \((\*)\) in the \( xt \)-plane: \(-\infty < t < \infty, -\infty < x < \infty\).

(b) Verify that if \( f \) and \( g \) are any twice-differentiable functions of a single real variable then
\[
 u(x, t) = f(x + ct) + g(x - ct)
\]
is a solution of \((\*)\) in the \( xt \)-plane.

Notes: 1. Solutions of the form (a) satisfy the initial/boundary conditions
\[(\text{B.C.)} \quad u(0, t) = u(L, t) \quad \text{for all } t \geq 0.\]
\[(\text{I.C.)} \quad \frac{\partial u}{\partial t} (x, 0) = 0 \quad \text{for all } 0 \leq x \leq L.\]

Solutions of the form (b) need not satisfy (B.C.) and (I.C.)

2. Solutions of the form (b) involve two "arbitrary" functions. In modeling physical phenomena, the PDE governing the evolution of the system must be supplemented with appropriate initial/boundary conditions to identify the "physically relevant" solutions among the vast number of possible solutions.

3. The wave operator \( \mathcal{L} = \frac{\partial^2}{\partial t^2} - \frac{c^2 \partial^2}{\partial x^2} \) in \((\*)\) is second-order and linear; i.e. \( \mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v) \) and \( \mathcal{L}(ku) = k \mathcal{L}(u) \). [Here \( u = u(x, t) \) and \( v = v(x, t) \) are arbitrary twice-differentiable functions of \( x \) and \( t \) and \( k \) is an arbitrary constant.] Therefore \((\*)\) is a linear homogeneous PDE: \( \mathcal{L}(u) = 0 \). A linear nonhomogeneous PDE has the form
\[
 \mathcal{L}(u) = f(x, t)
\]
where \( \mathcal{L} \) is a linear PD operator and \( f = f(x, t) \) is a specified function.
Superposition Principle: If \( u_1, u_2, \ldots, u_n \) are solutions to a linear homogeneous PDE \( L(u) = 0 \) and \( k_1, k_2, \ldots, k_n \) are any constants then the linear combination of solutions
\[
    u(x, t) = k_1 u_1(x, t) + k_2 u_2(x, t) + \ldots + k_n u_n(x, t)
\]
is also a solution to \( L(u) = 0 \).

For example,
\[
    u(x, t) = \sum_{j=1}^{n} k_j \cos\left(\frac{j\pi x}{L}\right) \sin\left(\frac{j\pi ct}{L}\right)
\]
\[
    = k_1 \cos\left(\frac{\pi ct}{L}\right) \sin\left(\frac{\pi x}{L}\right) + \ldots + k_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)
\]
solves \( \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \) for any integer \( n \geq 1 \) and any constants \( k_1, \ldots, k_n \).

Ex. 2] Determine whether or not the dispersive wave equation

\[
(\text{3rd order}) \quad \frac{\partial u}{\partial t} + (u \frac{\partial u}{\partial x}) + \frac{\partial^3 u}{\partial x^3} = 0
\]
is linear. (Korteweg–de Vries equation; cf. Sec 14.2, pp. 367–374.)

Ex. 3] (#10, p.5) Show that the solutions of the ODE

\[
(\dagger) \quad u''' - 3u'' + 4u = 0
\]
form a vector space. Find a basis for the solution space of \((\dagger)\).

By IV and FACT A on p.2 of "Vector Spaces" handout, it suffices to show that if \( L(u_1) = 0 \) and \( L(u_2) = 0 \) then \( L(c_1 u_1 + c_2 u_2) = 0 \) for any constants \( c_1 \) and \( c_2 \). (Check this)
\[ \{ e^{-t}, e^{2t}, t e^{2t} \} \] is a basis. Note: \( 0 = m^3 - 3m^2 + 4 = (m+1)(m^2-4m+4) = (m+1)(m-2)^2. \)