Review for Sec. 1.2

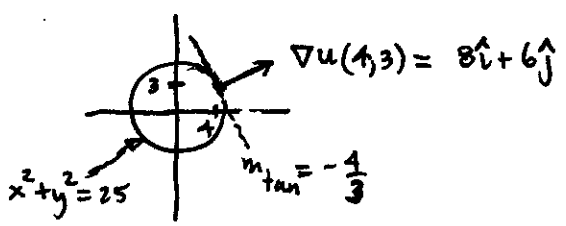
If $u = u(x,y)$ is a continuously differentiable (C') real function then the gradient of u is the vector function

$$\nabla u = \left< \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right> = \hat{\imath}\, u_x + \hat{\jmath}\, u_y .$$

FACTS: ① ∇u is orthogonal to the level curve $u(x,y) = \text{constant}$ at each point.

② ∇u points in the direction of maximum increase in u at each point.

Example: If $u(x,y) = x^2+y^2$ then $\nabla u = \langle 2x, 2y \rangle = \hat{\imath}\, 2x + \hat{\jmath}\, 2y$.



The directional derivative of $u = u(x,y)$ in the direction of the unit vector $\langle a, b \rangle$ is

$$D_{\langle a,b \rangle} u = \langle a, b \rangle \cdot \nabla u$$

This gives the rate of change of u in the direction of $\langle a, b \rangle$.

Example: If $u(x,y) = x^2+y^2$ then the directional derivative of u in the direction of $\langle 0,1 \rangle$ at the point $(4,3)$ is

$$D_{\langle 0,1 \rangle} u\,(4,3) = \langle 0,1 \rangle \cdot \nabla u(4,3) = \langle 0,1 \rangle \cdot \langle 8,6 \rangle = 6 .$$

Sec. 1.2 First-Order Linear PDEs

(Write the ~~four~~ three optional problems for Sec. 1.2 on board.)

A first-order linear PDE in two independent variables x and y has the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

where the coefficient functions a, b, c, and f are given.

Examples:

(a) $3u_x + 4u_y = 0$

(b) $yu_x + xu_y = 0$

(c) $u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2$

Ex 1 Solve (a) by the geometric method (or method of characteristics).

(a) $3u_x + 4u_y = 0 \iff \frac{3}{5}u_x + \frac{4}{5}u_y = 0$

$$D_{\langle \frac{3}{5}, \frac{4}{5}\rangle} u = \langle \tfrac{3}{5}, \tfrac{4}{5}\rangle \cdot \nabla u = 0$$

Geometrically, this says that the solution $u = u(x,y)$ is constant along any curve whose tangent remains parallel to $\langle \frac{3}{5}, \frac{4}{5}\rangle$; i.e. along any curve satisfying

$$\frac{dy}{dx} = \frac{4/5}{3/5} \Rightarrow y = \frac{4}{3}x + c \leftarrow \text{(A characteristic curve of the PDE (a))}$$

Therefore, along any line $y = \frac{4}{3}x + c$,

$$u(x,y) = u(x, \tfrac{4}{3}x + c) = u(0, \tfrac{4}{3}0 + c) = g(c)$$

Thus the general solution of (a) is

$$\boxed{u(x,y) = g(y - \tfrac{4}{3}x)}$$

where g is an arbitrary differentiable function of a single real variable.

Principle: The characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ satisfy $y' = \dfrac{b(x,y)}{a(x,y)}$.

Suppose $u = u(x,y) = \text{constant}$ at all points (x,y) on a smooth curve $y = y(x)$ in the plane. Suppose, furthermore, that

$$y'(x) = \frac{b(x,y)}{a(x,y)}$$

is the DE satisfied by $y = y(x)$. If u is C^1 then

$$\frac{d}{dx}\Big(u(x, y(x)) \Big) = \frac{d}{dx}\big(\text{constant}\big) = 0$$

$$\iff \quad u_x(x, y(x)) \cdot 1 + u_y(x, y(x))\, y'(x) = 0$$

$$\iff \quad u_x(x, y(x)) + u_y(x, y(x)) \frac{b(x, y(x))}{a(x, y(x))} = 0$$

$$\iff \quad a(x, y(x))\, u_x(x, y(x)) + b(x, y(x))\, u_y(x, y(x)) = 0$$

$$\iff \quad \langle a(x, y(x)), b(x, y(x)) \rangle \cdot \nabla u = 0$$

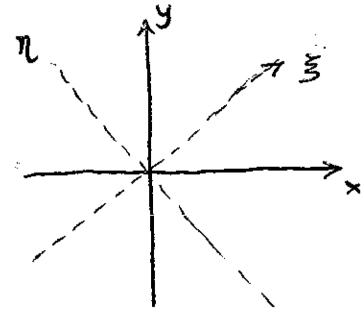
Conversely, suppose $\langle a(x,y), b(x,y) \rangle \cdot \nabla u = 0$. The above string of equivalent equations shows that $\frac{d}{dx}\Big[u(x, y(x)) \Big] = 0$ where $y = y(x)$ is a solution of the ODE $y'(x) = \dfrac{b(x,y)}{a(x,y)}$. That is, along such a curve $u(x,y) = \text{constant}$.

Conclusion: The characteristic curves of $a(x,y)u_x + b(x,y)u_y = 0$ satisfy

$$y' = \frac{b(x,y)}{a(x,y)}.$$

Ex 2 Solve (a) by the change-of-coordinates method.

(a) $3u_x + 4u_y = 0$



Let $\xi = 3x + 4y$

and $\eta = 4x - 3y$.

Then the chain rule gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = 3u_\xi + 4u_\eta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = 4u_\xi - 3u_\eta$$

Substituting into (a) yields

$$0 = 3u_x + 4u_y = 3(3u_\xi + 4u_\eta) + 4(4u_\xi - 3u_\eta) = 25u_\xi$$

Therefore u is independent of ξ, i.e. $u = f(\eta)$. The general solution of (a), expressed as a function of x and y, is

$$\boxed{u(x,y) = f(4x - 3y)} = f\left((-3)\left(y - \tfrac{4}{3}x\right)\right) = g\left(y - \tfrac{4}{3}x\right)$$

[where $g(t) = f(-3t)$]

Ex 3 Solve (b) $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$ subject to $u(x,0) = x^4$ for all real x.

Since the coefficients are variable, we cannot use the change-of-coordinates method. Therefore we use the geometric (or characteristics) method. Along each characteristic curve, the solution u is constant.

Characteristic curves: $y' = \frac{b(x,y)}{a(x,y)} = \frac{x}{y} \iff \int y\,dy = \int x\,dx$

$$\frac{y^2}{2} = \frac{x^2}{2} + \tilde{c}$$

$$y^2 - x^2 = c$$

hyperbolas! (or lines if $c=0$)

Along any such hyperbola, u is constant:

$$u(x,y) = u(x, \pm\sqrt{c+x^2}) = u(0, \pm\sqrt{c+0^2}) = f(c).$$

Therefore the general solution of (b) is

$$u(x,y) = f(y^2 - x^2).$$

We must determine f in order to satisfy the auxiliary condition $u(x,0) = x^4$ for all real x.

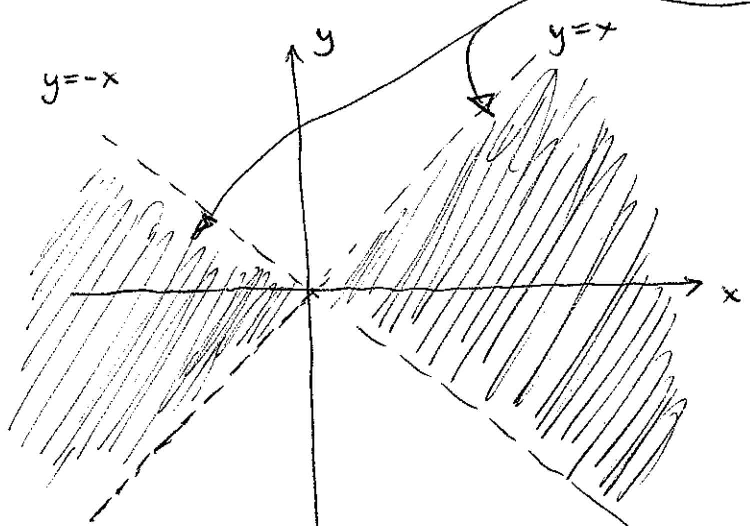
$$x^4 = u(x,0) = f(0^2 - x^2) = f(-x^2).$$

$$(-x^2)^2 = f(-x^2)$$

$$\therefore \quad f(t) = t^2 \quad \text{for all} \quad t \le 0.$$

Particular solution to I.V.P. $\boxed{u(x,y) = (y^2 - x^2)^2}$

This solution is valid (and uniquely determined) in the region where $y^2 - x^2 \le 0$.



Ex 4 Find the general solution of (c) $u_x + 2u_y + (2x-y)u = \overbrace{2x^2 + 3xy - 2y^2}^{(2x-y)(x+2y)}$.

Since $a(x,y)=1$ and $b(x,y)=2$ are constant, we can use the change-of-coordinates method.

Let $\xi = x+2y$, $\eta = 2x-y$. Then $u_x = u_\xi + 2u_\eta$ and $u_y = 2u_\xi - u_\eta$. Substituting into (c) gives

$$\eta\xi = (2x-y)(x+2y) = u_\xi + 2u_\eta + 2(2u_\xi - u_\eta) + \eta u = 5u_\xi + \eta u$$

$$\Rightarrow u_\xi + \tfrac{1}{5}\eta u = \tfrac{1}{5}\eta\xi. \qquad \text{Integrating factor: } e^{\int \frac{1}{5}\eta\, d\xi} = e^{\frac{1}{5}\eta\xi}.$$

For fixed η this is a linear 1st-order ODE in ξ.

$$e^{\frac{1}{5}\eta\xi}u_\xi + \tfrac{1}{5}\eta e^{\frac{1}{5}\eta\xi}u = \tfrac{1}{5}\eta\xi e^{\frac{1}{5}\eta\xi}$$

$$\frac{\partial}{\partial\xi}\left(e^{\frac{1}{5}\eta\xi}u\right) = \tfrac{1}{5}\eta\xi e^{\frac{1}{5}\eta\xi}$$

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$$e^{\frac{1}{5}\eta\xi}u = \int \tfrac{1}{5}\eta\xi e^{\frac{1}{5}\eta\xi}\, d\xi$$

$$\left\{ \begin{array}{ll} U = \xi & dV = \frac{1}{5}\eta e^{\frac{1}{5}\eta\xi} d\xi \\ & \text{Then} \\ dU = d\xi & V = e^{\frac{1}{5}\eta\xi} \end{array} \right.$$

$$= \xi e^{\frac{1}{5}\eta\xi} - \int e^{\frac{1}{5}\eta\xi}\, d\xi$$

$$= \xi e^{\frac{1}{5}\eta\xi} - \frac{5}{\eta}e^{\frac{1}{5}\eta\xi} + g(\eta)$$

"Constant" of integration may vary with the parameter η.

$$\therefore\ u = \xi - \frac{5}{\eta} + g(\eta)e^{-\frac{1}{5}\eta\xi}$$

$$\therefore\ \boxed{u(x,y) = x+2y - \frac{5}{2x-y} + g(2x-y)e^{\frac{2y^2-3xy-2x^2}{5}}}$$

where g is an arbitrary C^1-function of a single real variable.

Optional Problems for Sec. 1.2

~~A. Find the solution of~~

$$\cancel{x u_x + y u u_y = -xy}$$

~~satisfying the condition $u = 5$ on $xy = 1$. Find and sketch the region of the xy-plane in which your solution is valid.~~

A. ~~B.~~ (Cf. #9, p.10) Solve $a u_x + b u_y = x^2 + y^2$ where a and b are constants.

Challenge problem –

B. ~~C.~~ (2003 Master's Comprehensive Exam in Math 325) Solve

$$(*) \qquad x u_x + y u u_y = -xy$$

subject to the condition $u = 5$ on the hyperbola $xy = 1$. Find and sketch the largest region in the xy-plane in which the solution exists and is unique.

Hint on ~~C~~ B: Along a characteristic curve of (*) given parametrically by

$$x = x(s), \quad y = y(s),$$

we have by the chain rule that $\dfrac{du}{ds} = \dfrac{\partial u}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial u}{\partial y}\dfrac{dy}{ds}$.