Sec. 1.5 Well-Posed Problems

(Jacques Hadamard introduced the following notion.)

Well-posed PDE problems have the following properties:

1. Solutions exist.
2. Solutions are unique.
3. Solutions are stable (as a function of the "data" of the problem).

For example, we will see later that the following model for the vibrating string with its endpoints held fixed is well-posed:

\[
\begin{align*}
&u_t(x,t) - c^2 u_{xx}(x,t) = 0 \quad \text{for } 0 < x < L, \quad 0 < t, \\
&u(0,t) = 0 = u(L,t) \quad \text{for } t \geq 0 \\
&u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for } 0 \leq x \leq L.
\end{align*}
\]

Diffusion backwards in time is an example of a problem that is not well-posed (cf. p. 26).
Consider the Neumann problem

\begin{align*}
(1) & \quad \nabla^2 u = f(x,y,z) \quad \text{in } D \quad \nabla u \cdot \mathbf{n}(x,y,z) = 0 \\
(2) & \quad \frac{du}{dn} = 0 \quad \text{on } \partial D.
\end{align*}

(a) What can we surely add to any solution to get another solution? (So we don't have uniqueness.)

(b) Use the divergence theorem and the PDE (1) to show that

\[ \iiint_D f(x,y,z) \, dV = 0 \]

is a necessary condition for the Neumann problem (1)-(2) to have a solution.

(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

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Solution:

(a) If \( u_1 = u_1(x,y,z) \) is a solution to the Neumann problem (1)-(2),

then so is \( u_2 = u_1(x,y,z) + c \) \( \text{where } c \text{ is any constant}. \)

Thus (1)-(2) is not a well-posed problem because solutions are not unique. (Cf. #112(b), p. 168, however which shows that the difference between any two solutions of (1)-(2) must be a constant function on \( D \).)

(b) Suppose that \( u = u(x,y,z) \) is a solution to (1)-(2). Then
\[ 0 = \iint_D \frac{\partial u}{\partial n} \, dS \]

\[ = \iint_D \nabla u \cdot \hat{n} \, dS \]

\[ = \iiint_D \nabla \cdot (\nabla u) \, dV \quad \text{(Gauss' Divergence Theorem)} \]

\[ = \iiint_D \nabla^2 u \, dV \]

\[ = \iiint_D f(x,y,z) \, dV \]

(c) Physical interpretation of part (b): The presence of \( f \) in (1) indicates sources (+) and/or sinks (−) of heat energy. The condition (2) corresponds to an insulated boundary, i.e., no heat energy flow across \( \partial D \). (The system in \( D \) is “isolated” or “closed”.) In order for solutions \( u = u(x,y,z) \) to exist for the steady-state temperature distribution problem (1)-(2), the average value of the source/sink term \( f \) must be zero so that the total heat energy of the closed system in \( D \) is conserved.