Sec. 1.5 Well-Posed Problems

(Jacque Hadamard introduced the following notion.)

Well-posed PDE problems have the following properties:

1. Solutions exist.
2. Solutions are unique.
3. Solutions are stable (as a function of the "data" of the problem).

For example, we will see later that the following model for the undamped vibrating string with its endpoints held fixed is well-posed.

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{for } 0<x<L,\ 0<t, \\ u(0,t) = 0 = u(L,t) & \text{for } t \geq 0 \\ u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for } 0 \leq x \leq L. \end{cases}$$

Diffusion backwards in time is an example of a problem that is not well-posed (cf. p. 26).

(Example)

#4, p. 27

Consider the Neumann problem

$$(1) \qquad \nabla^2 u = f(x,y,z) \quad \text{in } D$$

$$(2) \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.$$

$\nabla u \cdot \vec{n}(x,y,z) = 0$

(a) What can we surely add to any solution to get another solution? (So we don't have uniqueness.)

(b) Use the divergence theorem and the PDE (1) to show that

$$\iiint_D f(x,y,z)\,dV = 0$$

is a necessary condition for the Neumann problem (1)-(2) to have a solution.

(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

Solution:

(a) If $u_1 = u_1(x,y,z)$ is a solution to the Neumann problem (1)-(2) then so is $u_2 = u_1(x,y,z) + c$ where c is any constant. Thus (1)-(2) is not a well-posed problem because solutions are not unique. (Cf. #12(b), p.168, however which shows that the difference between any two solutions of (1)-(2) must be a constant function on D.)

(b) Suppose that $u = u(x,y,z)$ is a solution to (1)-(2). Then

$$0 = \iint_{\partial D} \frac{\partial u}{\partial n}\, dS$$

$$= \iint_{\partial D} \nabla u \cdot \vec{n}\, dS$$

Gauss' Divergence Thm (p. 393)

$$= \iiint_{D} \nabla \cdot (\nabla u)\, dV$$

$$= \iiint_{D} \nabla^2 u\, dV$$

$$= \iiint_{D} f(x,y,z)\, dV$$

(c) Physical interpretation of part (b): The presence of f in (1) indicates sources (+) and/or sinks (−) of heat energy. The condition (2) corresponds to an insulated boundary, i.e. no heat energy flow across ∂D. (The system in D is "isolated" or "closed".) In order for solutions $u = u(x,y,z)$ to exist for the steady-state temperature distribution problem (1)–(2), the average value of the source/sink term f must be zero so that the total heat energy of the closed system in D is conserved.