Sec. 1.6 Types of Second-Order PDE's

Theorem: Let A, B, C, D, E, and F be real constants with $A^2+B^2+C^2>0$ and let $G=G(x,y)$ be a specified function. Consider

$$(*) \qquad Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G(x,y).$$

By introduction of new independent variables ξ and η and dependent variable ω, (*) can be transformed into one and only one of the following equivalent forms:

$$\frac{\partial^2\omega}{\partial\xi^2}-\frac{\partial^2\omega}{\partial\eta^2}+\gamma\omega=\varphi(\xi,\eta)$$ hyperbolic; $B^2-4AC>0$

$$\frac{\partial^2\omega}{\partial\xi^2}-\frac{\partial\omega}{\partial\eta}=\varphi(\xi,\eta)$$ parabolic; $B^2-4AC=0$

$$\frac{\partial^2\omega}{\partial\xi^2}+\frac{\partial^2\omega}{\partial\eta^2}+\gamma\omega=\varphi(\xi,\eta)$$ elliptic; $B^2-4AC<0$

$$\frac{\partial^2\omega}{\partial\xi^2}+\gamma\omega=\varphi(\xi,\eta)$$ parabolic degenerate case; reduces to solving an ODE; $B^2-4AC=0$

where γ is a constant taking one of the values $-1, 0,$ or 1.

Proof: See Elementary PDEs by Berg and McGregor (Holden-Day, 1966), pp. 23-28. For some of the techniques used in the proof, see (optional) HW ex. #5, p. 31.

To here Day 7 plus ... (see next pages)

Ex 1 Classify the types of the following PDEs, and find the general solution in the xy-plane whenever possible.

(a) $u_{xx}+u_{xy}-20u_{yy}=0.$ $\quad B^2-4AC=1^2-4(1)(-20)>0$; hyperbolic

(b) $3u_{xx}+4u_{yy}-u=0.$ $\quad B^2-4AC=0^2-4(3)(4)<0$; elliptic

(c) $u_{xx}+u_{yy}-2u_{xy}+4u=0.$ $\quad B^2-4AC=(-2)^2-4(1)(1)=0$; parabolic

The Hyperbolic Case

Consider

(1)
$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)\left(\gamma\frac{\partial}{\partial x}+\delta\frac{\partial}{\partial y}\right)u=0$$

in the plane $-\infty<x<\infty,\ -\infty<y<\infty$, where $\alpha\delta-\beta\gamma\neq 0$. Let

$$\xi=\beta x-\alpha y,$$
$$\eta=\delta x-\gamma y.$$

Then the general C^2 – solution to (1) in the plane is $u=f(\xi)+g(\eta)$ where f and g are arbitrary C^2 – functions of a single real variable.

Proof: A simple computation shows that

$$x=\frac{\alpha\eta-\gamma\xi}{\alpha\delta-\beta\gamma}\ \text{ and }\ y=\frac{\beta\eta-\delta\xi}{\alpha\delta-\beta\gamma}.$$

If $v=v(x,y)$ is any C^1 – function in the plane, then

$$\frac{\partial v}{\partial \xi}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \xi}=\frac{1}{\alpha\delta-\beta\gamma}\left(-\gamma\frac{\partial v}{\partial x}-\delta\frac{\partial v}{\partial y}\right),$$

$$\frac{\partial v}{\partial \eta}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \eta}=\frac{1}{\alpha\delta-\beta\gamma}\left(\alpha\frac{\partial v}{\partial x}+\beta\frac{\partial v}{\partial y}\right).$$

Consequently, (1) is equivalent to

$$\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right)=0.$$

Integrating with respect to η holding ξ fixed gives

$$\frac{\partial u}{\partial\xi}=c_1(\xi).$$

Integrating this last equation with respect to ξ holding η fixed gives

$$u=\int c_1(\xi)d\xi+c_2(\eta).$$

Thus $u=f(\xi)+g(\eta)$ and, since u is a C^2 – function in the plane, it follows that f and g are C^2 – functions of a single real variable.

Conversely, if $u(x,y)=f(\beta x-\alpha y)+g(\delta x-\gamma y)$ where f and g are C^2 – functions of a single real variable, then clearly u is a C^2 – function in the plane. Using the chain rule, one easily verifies that u solves (1) in the plane.

The Parabolic Case

Consider

(2) $$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)^2 u=0$$

in the plane $-\infty<x<\infty,\ -\infty<y<\infty$, where $\alpha^2+\beta^2>0$. Let

$$\xi=\beta x-\alpha y,$$
$$\eta=\alpha x+\beta y.$$

Then the general C^2 – solution to (2) in the plane is $u=\eta f(\xi)+g(\xi)$ where f and g are arbitrary C^2 – functions of a single real variable.

Proof: Homework exercise.

(a) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} - 20\frac{\partial^2}{\partial y^2}\right) u = 0$

$$\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right) u = 0$$

(This is the hyperbolic case on the handout with $\alpha = 1, \beta = 5, \gamma = 1, \delta = -4$.)

Let $\begin{cases} \xi = 5x - y \\ \eta = +4x + y \end{cases}$

Then the chain rule implies that, as operators,

$$\frac{\partial}{\partial x} = 5\frac{\partial}{\partial \xi} + 4\frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

Therefore

$$\frac{\partial}{\partial x} + 5\frac{\partial}{\partial y} = \left(5\frac{\partial}{\partial \xi} + 4\frac{\partial}{\partial \eta}\right) + 5\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 9\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y} = \left(5\frac{\partial}{\partial \xi} + 4\frac{\partial}{\partial \eta}\right) - 4\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 9\frac{\partial}{\partial \xi}$$

so the PDE (a) is equivalent to

$$\left(\cancel{9}\frac{\partial}{\partial \eta}\right)\left(\cancel{9}\frac{\partial}{\partial \xi}\right) u = 0.$$

Thus $\qquad \frac{\partial u}{\partial \xi} = c_1(\xi) \qquad$ (independent of η)

and $\qquad u = \int c_1(\xi)\,d\xi + c_2(\eta).$

I.e. $\qquad u = f(\xi) + g(\eta)$

or $\qquad \boxed{u(x,y) = f(5x-y) + g(4x+y)}$

where f and g are $\underbrace{}_{\text{arbitrary}}$ C^2-functions of a single real variable.

(c) $$\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right) u + 4u = 0$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u + 4u = 0$$

(This is the parabolic case on the handout with $\alpha = 1$, $\beta = -1$.)

Let $\xi = +x + y$

$\eta = x - y$

Then $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ so the PDE is equivalent to

$$\left(\cancel{2}\frac{\partial}{\partial \eta}\right)\left(\cancel{2}\frac{\partial}{\partial \eta}\right) u + \cancel{4}u = 0$$

$$\therefore \quad u = c_1(\xi)\cos(\eta) + c_2(\xi)\sin(\eta)$$

i.e. $$\boxed{u(x,y) = f(x+y)\cos(x-y) + g(x+y)\sin(x-y)}$$

where f, g are arbitrary C^2-functions of a single real variable.

Covered on Day 7

Note: For constant coefficient second order linear PDEs in more than two independent variables

Since mixed partial derivatives are equal, we may assume $a_{ij} = a_{ji}$

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = f(x_1, \dots, x_n) \qquad (n \geq 3)$$

the classification is more complicated and incomplete. The type depends on the signs of the eigenvalues of the (symmetric) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

See text, pp. 29-30.

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Section 1.6 Supplementary Problems

Classify the following second order, linear partial differential equations as either hyperbolic, elliptic, or parabolic. In the case of hyperbolic or parabolic equations, find the general solution in the plane.

A. $u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$

B. $u_{xx} - 3u_{xy} - 4u_{yy} = 0$

C. $u_{xx} + 9u_{yy} - 6u_{xy} = 0$

D. $u_{xx} + 4u_{yy} - 4u_{xy} + u = 0$

E. $u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} + 2u_x - 2u_y = 0$

F. $u_{xx} + 2u_{xy} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$

G. $u_{xx} + u_{yy} - 2u_{xy} + u_x - u_y = 0$

H. $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_y = 0$

I. $u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} = \sin(x + y)$