Chapter 2: Waves and Diffusions

Sec 2.1 The Wave Equation

Example (10, p.37) Solve $u_{xx} + u_{tt} = 0$ in the xt-plane subject to the initial conditions

$$u(x,0) = 25x^2 + 6x^3 \quad \text{and} \quad u_t(x,0) = -10x + 18x^2 \quad \text{for all } -\infty < x < \infty.$$  

Solution: From previous work, we know that the general solution to the PDE in the xt-plane is $u(x,t) = f(5x-t) + g(4x+t)$ where $f$ and $g$ are $C$-functions of a single real variable. We must determine $f$ and $g$ so the I.C.'s are satisfied.

(1) $25x^2 + 6x^3 = u(x,0) = f(5x) + g(4x)$ for all $-\infty < x < \infty.$

$$u_t(x,0) = -f'(5x-t) + g'(4x+t)$$

(2) $\begin{cases} -10x + 18x^2 = u_t(x,0) = -f'(5x) + g'(4x) & \text{for all } -\infty < x < \infty. \\ 50x + 192x^2 = 5f''(5x) + 4g''(4x) & \ldots \ldots \end{cases}$

Multiplying (2) by 5 and adding to (1') gives

$$432x^2 = 9g'(4x) \Rightarrow 48x^2 = g'(4x)$$

Letting $z = 4x$ gives $g'(z) = 3z^2$

$$\Rightarrow g(z) = \frac{z^3}{3} + c_1 \quad \text{for all } -\infty < z < \infty$$

Multiplying (2) by -4 and adding to (1') gives

$$90x = 9f'(5x) \Rightarrow 10x = f'(5x)$$

(Let $z = 5x$) \Rightarrow $f'(z) = 2z$

$$\Rightarrow f(z) = \frac{z^2}{2} + c_2 \quad \text{for all } -\infty < z < \infty.$$
\[ u(x,t) = f(5x-t) + g(4x+t) \]

\[ u(x,t) = (5x-t)^2 + c_1 + (4x+t)^3 + c_2 \]

However, the constants \( c_1 \) and \( c_2 \) are not independent. For example,

\[ 25x^2 + 64x^3 = u(x,0) = (5x)^2 + c_1 + (4x)^3 + c_2 \]

\[ = 25x^2 + c_1 + 64x^3 + c_2 \]

so we must \( c_1 + c_2 = 0 \). Therefore the particular solution to our problem is

\[ u(x,t) = (5x-t)^2 + (4x+t)^3 \]

Similar techniques give the following result:

The solution to \( u_{tt} - c^2 u_{xx} = 0 \) in the \( xt \)-plane satisfying the initial conditions

\[ u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \]

for all \(-\infty < x < \infty\)

is given by

\[ u(x,t) = \frac{1}{2} \left[ \varphi(x+ct) + \varphi(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds \]

(Here we assume that \( \varphi \) is a \( C^1 \)-function and \( \psi \) is a \( C^1 \)-function of a single real variable.)

Note: You are responsible for understanding this result on Exam I.
Ex 2] Solve $u_{tt} - u_{xx} = 0$ in the $xt$-plane subject to

\[ u(x, 0) = e^{-x^2} \text{ and } u_x(x, 0) = 0 \text{ for } -\infty < x < \infty. \]

\text{Solv: } u(x, t) = \frac{1}{2} \left[ e^{-\left(\frac{x-t}{2}\right)^2} + e^{-\left(\frac{x+t}{2}\right)^2} \right] + \frac{1}{2} \int_{-\infty}^{x+t} \int_{-\infty}^{x-t} \psi(d) \, ds

\hspace{2cm} = \frac{1}{2} e^{-\left(\frac{x-t}{2}\right)^2} + \frac{1}{2} e^{-\left(\frac{x+t}{2}\right)^2}

Note: The solution consists of two "traveling" waves, one that moves to the right along the $x$-axis with speed 1 and one that moves to the left with speed 1.

Ex 3] (#5, p.37. The Hammer Blow) Find the solution of $u_{tt} - c^2 u_{xx} = 0$

satisfying $u(x, 0) = q(x) = 0$ and $u_t(x, 0) = \Psi(x) = \begin{cases} \frac{1}{2} \Psi \text{ for } -a < x < a, \\ 0 \text{ otherwise}. \end{cases}$

Sketch the profile of the string at each of the successive instants

\[ t = \frac{a}{2c}, \frac{a}{2c}, \frac{3a}{2c}, \frac{2a}{c}, \text{ and } \frac{5a}{c}. \]

\text{Solv: } u(x, t) = \frac{1}{2} \left[ q(x-ct) + q(x+ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \Psi(s) \, ds

\hspace{4cm} = \frac{1}{2c} \int_{x-ct}^{x+ct} \Psi(s) \, ds

We will make use of the following facts to evaluate the integral.
0. If $A \in \mathbb{R}$ then $\chi_A(\xi) = \begin{cases} 1 & \text{if } \xi \in A, \\ 0 & \text{o.w.} \end{cases}$

1. $\int_c^d f(\xi) d\xi = \int_{-\infty}^{\infty} \chi_{(c,d)}(\xi) f(\xi) d\xi$

2. $\chi_A(\xi) \chi_B(\xi) = \chi_{A \cap B}(\xi)$ for all $A, B \in \mathbb{R}$ and all $\xi \in \mathbb{R}$.

3. $\int_{-\infty}^{\infty} \chi_{(c,d)}(\xi) d\xi = \text{length of } (c,d) \quad (= d - c)$.

$\therefore \quad u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \chi(\xi) d\xi = \frac{1}{2c} \int_{x-ct}^{x+ct} \chi_{(-a,a)}(\xi) d\xi$

$= \frac{1}{2c} \int_{-\infty}^{\infty} \chi_{(x-ct,x+ct)}(\xi) \chi_{(-a,a)}(\xi) d\xi$

$= \frac{1}{2c} \int_{-\infty}^{\infty} \chi_{(x-ct,x+ct) \cap (-a,a)}(\xi) d\xi$

$= \frac{1}{2c} \text{length of } \{(x-ct, x+ct) \cap (-a,a)\}$.

\[ t = \frac{a}{2c} \quad u(x, \frac{a}{2c}) = \frac{1}{2c} \text{length of } \{(x-\frac{a}{2}, x+\frac{a}{2}) \cap (-a,a)\} \]
\[ t = \frac{a}{c} \quad u(x, \frac{a}{c}) = \frac{1}{2c} \text{length of } \{(x-a, x+a) \cap (-a, a)\} \]

\[ t = \frac{3a}{2c} \quad u(x, \frac{3a}{2c}) = \frac{1}{2c} \text{length of } \{(x-\frac{3a}{2}, x+\frac{3a}{2}) \cap (-a, a)\} \]

\[ t = \frac{2a}{c} \quad u(x, \frac{2a}{c}) = \frac{1}{2c} \text{length of } \{(x-2a, x+2a) \cap (-a, a)\} \]

\[ t = \frac{5a}{2c} \quad u(x, \frac{5a}{2c}) = \frac{1}{2c} \text{length of } \{(x-\frac{5a}{2}, x+\frac{5a}{2}) \cap (-a, a)\} \]

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**Notes:**

1. The function \( u(x,t) = \frac{1}{2c} \text{length of } \{(x-ct, x+ct) \cap (-a, a)\} \) is called a "generalized solution" of the problem: \( u_{tt} - c^2 u_{xx} = 0 \) in \(-\infty < x < \infty, \ t > 0\), and \( u(x,0) = 0 \) and \( u_t(x,0) = x(\xi(x)) \) for \(-\infty < x < \infty\). Note that \( u = u(x,t) \) fails to be differentiable at all points on the lines \( x-ct = \pm a \) and \( x+ct = \pm a \), hence it is not a "classical solution."
of the problem (i.e. a function of $x$ and $t$ that:

(a) is $C^2$ and satisfies the PDE at all points of the region $-\infty < x < \infty$, $0 < t < \infty$;

(b) is $C^1$ on the closure of the region $-\infty < x < \infty$, $0 \leq t < \infty$;

and (c) satisfies the initial conditions on the boundary $t=0$ of the region.)

2. The singularities of the generalized solution (i.e. points where the function fails to be differentiable) propagate along those characteristics $|x \pm ct| = a$ of the PDE $u_{tt} - c^2 u_{xx} = 0$ which pass through the points $(\pm a, 0)$ where the initial data is singular.