Definition. For $n \geq 1$, let $C^n[a,b]$ denote the vector space of $n$ times continuously differentiable complex-valued functions on the interval $[a,b]$. Let $C[a,b]$ denote the vector space of continuous complex-valued functions on $[a,b]$, and let $V$ be a vector subspace of $C[a,b]$. A function $T$, defined on $V$, taking values in $C[a,b]$, and with the property that

$$T(c_1f_1 + c_2f_2) = c_1T(f_1) + c_2T(f_2)$$

for all numbers $c_1$ and $c_2$ and all $f_1$ and $f_2$ in $V$, is called a linear operator on $V$.

Example 1. The differential operator $T = -\frac{d^2}{dx^2}$ is a linear operator on $C^2[a,b]$.

Definition. A linear operator $T : V \rightarrow C[a,b]$ is called symmetric if $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f$ and $g$ in $V$.

Example 2. Show that the operator $T = -\frac{d^2}{dx^2}$ is symmetric on $V_D = \{ f \in C^2[0,\pi] : f(0) = f(\pi) \}$.

Solution: Let $f$ and $g$ belong to $V_D$. Then two integrations by parts and use of the boundary conditions $f(0) = f(\pi)$ and $g(0) = g(\pi)$ show that

$$\langle Tf, g \rangle = \int_0^\pi f(x) g''(x) dx - \int_0^\pi f''(x) g(x) dx = \left. f(x)g(x) \right|_{x=0}^{x=\pi} - \int_0^\pi f''(x) g(x) dx$$

$$= \int_0^\pi \left[ -f''(x)g''(x) \right] dx.$$ Therefore $\langle Tf, g \rangle = \langle f, Tg \rangle$ so $T$ is symmetric on $V_D$.

Homework A. Show that $T = -\frac{d^2}{dx^2}$ is symmetric on the following subspaces of $C[0,\pi]$.

1. $V_N = \{ f \in C^2[0,\pi] : f'(0) = 0 = f'(\pi) \}$.
2. $V_P = \{ f \in C^2[-\pi,\pi] : f(-\pi) = f(\pi), f'(-\pi) = f'(\pi) \}$.
3. $V_R = \{ f \in C^2[0,\pi] : f'(0) - a_0f(0) = 0 = f'(\pi) + a_\pi f(\pi) \}$.
   (Here $a_0$ and $a_\pi$ are fixed real constants.)

Example 3. Show that the operator $Tf(x) = (1-x^2)f''(x) - xf'(x)$ is symmetric on the vector space $V_T = \{ f \in C^2(-1,1) : f$ and $f'$ are bounded on $(-1,1) \}$ equipped with the inner product

$$(*) \quad \langle f, g \rangle = \int_{-1}^1 f(x)g(x)(1-x^2)^{-1/2} dx.$$ Solution: Let $f$ and $g$ belong to $V_T$. Then $\langle Tf, g \rangle = \int_{-1}^1 \left[ (1-x^2)f''(x) - xf'(x) \right] g(x)(1-x^2)^{-1/2} dx$
Solution (cont.):
\[
\langle Tf, g \rangle = \int_{-1}^{1} \left[ (1-x^2)^{\frac{1}{2}} f'(x) - x(1-x^2)^{\frac{1}{2}} f(x) \right] \overline{g(x)} \, dx = \int_{-1}^{1} \overline{g(x)} \frac{d}{dx} \left[ (1-x^2)^{\frac{1}{2}} f(x) \right] \, dx
\]
\[
= (1-x^2)^{\frac{1}{2}} f'(x) \overline{g(x)} \bigg|_{-1}^{1} - \int_{1}^{1} (1-x^2)^{\frac{1}{2}} f'(x) \overline{g(x)} \, dx.
\]
Since \( \lim_{x \to 1^-} (1-x^2)^{\frac{1}{2}} f'(x) \overline{g(x)} \) is bounded and \( \lim_{x \to 1^+} (1-x^2)^{\frac{1}{2}} f'(x) \overline{g(x)} = 0 \) by the Squeeze Theorem, it follows that \( (1-x^2)^{\frac{1}{2}} f'(x) \overline{g(x)} \) is bounded near \( x=1 \).

Therefore
\[
\langle Tf, g \rangle = -\int_{1}^{1} (1-x^2)^{\frac{1}{2}} g'(x) f(x) \, dx = -\int_{1}^{1} (1-x^2)^{\frac{1}{2}} \frac{g''(x)}{g'(x)} f(x) \, dx + \int_{1}^{1} f(x) \frac{d}{dx} \left[ (1-x^2)^{\frac{1}{2}} g'(x) \right] \, dx.
\]

But an argument similar to the one above shows that \( -\int_{1}^{1} (1-x^2)^{\frac{1}{2}} \frac{g''(x)}{g'(x)} f(x) \, dx = 0 \) so

\[
\langle Tf, g \rangle = \int_{-1}^{1} f(x) \left[ (1-x^2)^{\frac{1}{2}} g'(x) - x (1-x^2)^{\frac{1}{2}} g'(x) \right] \, dx = \int_{-1}^{1} f(x) \left[ (1-x^2)^{\frac{1}{2}} g''(x) - (1-x^2)^{\frac{1}{2}} x g'(x) \right] \, dx
\]

and hence \( \langle Tf, g \rangle = \langle f, Tg \rangle \). Consequently \( T \) is symmetric on \( V_T \).

---

**Definition.** Let \( T : V \to C[a,b] \) be a linear operator. If \( \lambda \) is a complex number and \( f \neq 0 \) is a function in \( V \) such that \( Tf = \lambda f \) then \( \lambda \) is called an **eigenvalue** of \( T \) and \( f \) is called an **eigenfunction** of \( T \).

**Example 4.** The operator \( T = \frac{d^2}{dx^2} \) on \( V_D = \{ f \in C^2[0,\pi] : f(0) = f(\pi) = 0 \} \) has eigenvalues \( \lambda_n = n^2 \) \((n = 1, 2, 3, \ldots)\) and corresponding eigenfunctions \( f_n(x) = \sin(nx) \) \((n = 1, 2, 3, \ldots)\).

**Theorem 2.** Let \( T : V \to C[a,b] \) be a symmetric operator. Then all the eigenvalues of \( T \) are real numbers.

**Note:** The proof of Theorem 2 will make use of the following properties of an inner product.

1. For all \( f \) in \( V \), \( \langle f, f \rangle \geq 0 \), with equality only if \( f = 0 \).
2. For all \( f, g, \) and \( h \) in \( V \), \( \langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle \).
3. For all \( f \) and \( g \) in \( V \) and all complex numbers \( \alpha \), \( \langle \alpha f, g \rangle = \alpha \langle f, g \rangle \).
4. For all \( f \) and \( g \) in \( V \), \( \langle f, g \rangle = \langle g, f \rangle \).

It is an easy consequence of (3) and (4) that \( \langle f, \alpha g \rangle = \overline{\alpha} \langle f, g \rangle \).

**Proof of Theorem 2:** Let \( \lambda \) be an eigenvalue of \( T \) and let \( f \) be a nonzero function in \( V \).
Proof of Theorem 2 (cont.): such that $Tf = \lambda f$. Then $\lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle Tf, f \rangle = \langle f, T \rangle = \lambda \langle f, f \rangle$, consequently, $\lambda = \lambda f$. But $f \neq 0$ so $\langle f, f \rangle > 0$ by (1).

Hence $\lambda - \lambda = 0$ or equivalently $\lambda = \lambda$. That is, $\lambda$ is a real number.

**Theorem 1.** Let $T : V \rightarrow C[a,b]$ be a symmetric operator. If $f_1$ and $f_2$ are eigenfunctions of $T$ corresponding to distinct eigenvalues $\lambda_1$ and $\lambda_2$ of $T$, then $f_1$ and $f_2$ are orthogonal on $[a,b]$.

**Proof:** Let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of $T$ on $V$. That is, $\lambda_1 \neq \lambda_2$ and there exist nonzero functions $f_1$ and $f_2$ in $V$ such that $Tf_1 = \lambda_1 f_1$ and $Tf_2 = \lambda_2 f_2$. Thus $\lambda_1 \langle f_1, f_2 \rangle = \langle \lambda_1 f_1, f_2 \rangle = \langle Tf_1, f_2 \rangle = \langle f_1, T f_2 \rangle = \langle f_1, \lambda_2 f_2 \rangle = \lambda_2 \langle f_1, f_2 \rangle$ by Theorem 2. Therefore $(\lambda_1 - \lambda_2) \langle f_1, f_2 \rangle = \lambda_1 \langle f_1, f_2 \rangle - \lambda_2 \langle f_1, f_2 \rangle = 0$.

But $\lambda_1 - \lambda_2 \neq 0$ so $\langle f_1, f_2 \rangle = 0$. That is, $f_1$ and $f_2$ are orthogonal on $[a,b]$.

**Example 5.** Let $Tf(x) = (1 - x^2) f''(x) - xf'(x)$ be the operator on the vector space $V_T = \{ f \in C^2(-1,1) : f$ and $f'$ are bounded on $(-1,1) \}$ equipped with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) \overline{g(x)} (1 - x^2)^{-1/2} \, dx.$$  

Show that all the eigenvalues of $T$ are real numbers and the eigenfunctions of $T$ corresponding to distinct eigenvalues are orthogonal on the interval $(-1,1)$ relative to the inner product $(\ast)$.

**Solution:** Example 3 shows that the operator $T$ is symmetric on $V_T$, equipped with the inner product $(\ast)$. According to Theorem 2, all the eigenvalues of $T$ are real numbers. By Theorem 1, eigenfunctions of $T$ corresponding to distinct eigenvalues are orthogonal on $(-1,1)$ relative to the inner product $(\ast)$.

**Note:** It can be shown that the symmetric operator $T$ in Example 5 has eigenvalues $\lambda_n = -n^2$ ($n = 0, 1, 2, \ldots$) and corresponding eigenfunctions that are the Tchebicheff polynomials: $f_n(x) = \cos \left( n \cos^{-1}(x) \right)$ ($n = 0, 1, 2, \ldots$). The third three Tchebicheff polynomials are $f_0(x) = 1$, $f_1(x) = x$, and $f_2(x) = 2x^2 - 1$. The Tchebicheff polynomials are solutions to Tchebicheff’s differential equation $(1 - x^2) f''(x) - xf'(x) = \lambda f(x)$ on the interval $(-1,1)$ with $\lambda = \lambda_n = -n^2$.  

Note on #1 of “Additional Problems for Section 5.3”:

Consider the operator

\[ Tf(r) = \frac{1}{r} \frac{d}{dr} \left( rf'(r) \right) - \frac{n^2}{r^2} f(r) \quad (0 < r \leq 1) \]

on the domain

\[ V_B = \{ f \in C^2[0,1]: f(1) = 0 \text{ and } f, f' \text{ are bounded on } (0,1) \} \]

The inner product

\[ \langle f, g \rangle = \int_0^1 f(r) \overline{g(r)} r dr \]

on \( V_B \) arises naturally from the inner product

\[ \langle h, k \rangle = \int_0^{2\pi} \int_0^1 h(r, \theta) \overline{k(r, \theta)} r dr d\theta \]

for square-integrable functions \( h \) and \( k \) in the unit disk \( D = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \} \) of the plane.

The eigenvalue equation \( Tf = \lambda f \) for this operator is equivalent to Bessel’s equation of order \( n \) (cf. pp. 252 and 268 in Strauss):

\[ \frac{1}{r} \frac{d}{dr} \left( rf''(r) \right) - \frac{n^2}{r^2} f(r) = \lambda f(r). \]

For applications of #1 in solving PDEs see Strauss, Section 10.2: Vibrations of a (Circular) Drumhead.

**Theorem 3.** Let \( T = -\frac{d^2}{dx^2} \) be symmetric on a vector subspace \( V \) of \( C^2[a,b] \) which is closed under the operation of complex conjugation of functions. If \( f(b)f'(b) - f(a)f'(a) \leq 0 \) for all real-valued functions \( f \) in \( V \) then \( T \) has no negative eigenvalues.

Proof: Let \( \lambda \) be an eigenvalue of \( T \) on \( V \) and let \( f \) be an eigenfunction of \( T \) on \( V \) corresponding to \( \lambda \); that is, \( f \) is a nonzero function in \( V \) such that \( Tf = \lambda f \). This last condition is equivalent to

\[ f''(x) + \lambda f(x) = 0 \quad \text{for all } x \in [a,b]. \]

Take the complex conjugate of this identity, and use the fact that \( \lambda \) is a real number (cf. Theorem 2) to obtain \( \overline{f''(x)} + \lambda \overline{f(x)} = 0 \) for all \( x \in [a,b] \). Thus \( T \overline{f} = \lambda \overline{f} \), so \( \overline{f} \) is an eigenfunction of \( T \) on \( V \) corresponding to \( \lambda \). Consequently, at least one of the functions

\[ \phi = \text{Re}(f) = \frac{1}{2} \left( f + \overline{f} \right) \quad \text{or} \quad \psi = \text{Im}(f) = \frac{1}{2i} \left( f - \overline{f} \right) \]

is not the zero function and hence is a real-valued eigenfunction of \( T \) on \( V \) corresponding to \( \lambda \).

Suppose for the sake of argument that \( \phi \neq 0 \). Observe that

\[ \lambda \langle \phi, \phi \rangle = \langle T \phi, \phi \rangle = \int_a^b -\phi''(x) \overline{\phi(x)} dx = \int_a^b -\phi''(x) \phi(x) dx = -\phi(b) \overline{\phi(b)} + \phi'(a) \overline{\phi(a)} + \int_a^b (\phi'(x))^2 dx. \]

Since the last member of this identity is nonnegative and \( \langle \phi, \phi \rangle \) is positive, it follows that \( \lambda \geq 0 \). Q.E.D.

**Example 7.** All the eigenvalues of \( T = -\frac{d^2}{dx^2} \) on \( V_D = \{ f \in C^2[0,\pi]: f(0) = f(\pi) = 0 \} \) satisfy \( \lambda \geq 0 \).

**Note:** You will need to generalize Theorem 3 and its proof in order to work #1(c) on “Additional Problems for Section 5.3”.
1. (a) Let \( n \) be a nonnegative integer. Show that the operator \( T \) given by
\[
Tf(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{df}{dr} \right) - \frac{n^2}{r^2} f(r) \quad (0 < r \leq 1)
\]
is symmetric on the vector space
\[
V_B = \{ f \in C^2(0,1] : f(1) = 0, f \text{ and } f' \text{ bounded on } (0,1] \}
\
equipped with the inner product
\[
\langle f, g \rangle = \int_0^1 f(r)g(r)r dr.
\]

(b) Show that the eigenvalues of \( T \) on \( V_B \) are real numbers.

(c) Are the eigenvalues of \( T \) on \( V_B \) positive? Justify your answer.

(d) Are the eigenfunctions of \( T \) on \( V_B \), corresponding to distinct eigenvalues, orthogonal on \( (0,1) \) relative to the inner product (*)? Justify your answer.

2. Use separation of variables to solve the variable density vibrating string problem:
\[
\frac{1}{(1+x)^2} u_{tt} - u_{xx} = 0 \quad \text{for } 0 < x < 1, \ 0 < t < \infty,
\]
\[
u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 \leq t < \infty,
\]
\[
u(x,0) = x(1-x)\sqrt{1+x} \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for } 0 \leq x \leq 1.
\]

Hints on 2: (a) Show that the operator \( T \) given by \( Tf(x) = -(1+x)^2 f''(x) \) is symmetric on
\[
V_B = \{ f \in C^2[0,1] : f(0) = 0 = f(1) \}, \text{ equipped with the inner product } \langle f, g \rangle = \int_0^1 f(x)g(x)(1+x)^{-2} dx.
\]

Conclude that all the eigenvalues \( \lambda \) of the problem \( X''(x) + \frac{\lambda}{(1+x)^2} X(x) = 0, \ X(0) = 0 = X(1) \) are real.

(b) Show that (nearly) all solutions to \( X''(x) + \frac{\lambda}{(1+x)^2} X(x) = 0 \) on \( (0,1) \) are of the form \( X(x) = (1+x)^\gamma \)
where \( \alpha \) is an appropriately chosen (possibly complex) constant. Explicitly, show that the general solution is:
\[
X(x) = (1+x)^{1/2} \left[ c_1 \left( 1+x \right)^{\frac{\sqrt{4\lambda-1}}{2}} + c_2 \left( 1+x \right)^{-\frac{\sqrt{4\lambda-1}}{2}} \right] \quad \text{if } 1-4\lambda > 0,
\]
\[
X(x) = (1+x)^{1/2} \left[ c_1 + c_2 \ln(1+x) \right] \quad \text{if } 1-4\lambda = 0,
\]
\[
X(x) = (1+x)^{1/2} \left[ c_1 \cos \left( \frac{\sqrt{4\lambda-1}}{2} \ln(1+x) \right) + c_2 \sin \left( \frac{\sqrt{4\lambda-1}}{2} \ln(1+x) \right) \right] \quad \text{if } 1-4\lambda < 0.
\]