Mathematics 325
Lecture Notes for Section 5.4: Completeness

Definition. Any pair of boundary conditions

$$(*) \qquad \begin{cases} \alpha_1 f(a)+\beta_1 f(b)+\gamma_1 f'(a)+\delta_1 f'(b)=0 \\ \alpha_2 f(a)+\beta_2 f(b)+\gamma_2 f'(a)+\delta_2 f'(b)=0 \end{cases}$$

where α_j, β_j, γ_j, and δ_j $(j=1,2)$ are real constants, is called ***symmetric*** for the operator $T=-\dfrac{d^2}{dx^2}$ if

$$\langle Tf_1,f_2\rangle-\langle f_1,Tf_2\rangle=\left[f_1(x)\overline{f_2'(x)}-f_1'(x)\overline{f_2(x)}\right]\Big|_{x=a}^{x=b}=0$$

for all f_1 and f_2 in $C^2[a,b]$ satisfying (*).

Example 1. The homogeneous Dirichlet boundary conditions are of the form (*):

$$0=f(a)=1f(a)+0f(b)+0f'(a)+0f'(b)=0 \qquad (\alpha_1=1,\ \beta_1=0,\ \gamma_1=0,\ \delta_1=0)$$
$$0=f(b)=0f(a)+1f(b)+0f'(a)+0f'(b)=0 \qquad (\alpha_2=0,\ \beta_2=1,\ \gamma_2=0,\ \delta_2=0).$$

Notes: (a) In a similar way, it is easy to see that the homogeneous Neumann, periodic, and Robin boundary conditions can also be written in the form (*).

(b) For all f_1 and f_2 in $V_D=\{f\in C^2[a,b]: f(a)=0=f(b)\}$ we have

$$\left[f_1(x)\overline{f_2'(x)}-f_1'(x)\overline{f_2(x)}\right]\Big|_{x=a}^{x=b}=0$$

so the homogeneous Dirichlet boundary conditions are symmetric for the operator $T=-\dfrac{d^2}{dx^2}$.

Homework A. Show that the following boundary conditions are symmetric for the operator $T=-\dfrac{d^2}{dx^2}$:

(1) homogeneous Neumann boundary conditions: $f'(a)=0=f'(b)$;
(2) homogeneous periodic boundary conditions: $f(a)=f(b)$ and $f'(a)=f'(b)$;
(3) homogeneous Robin boundary conditions: $f'(a)-\alpha f(a)=0=f'(b)+\beta f(b)$.

<u>**Theorem 1.**</u> Consider the eigenvalue problem for the operator $T=-\dfrac{d^2}{dx^2}$:

$$-X''(x)=\lambda X(x) \text{ in } (a,b)$$

subject to symmetric boundary conditions

$$\begin{cases} \alpha_1 f(a)+\beta_1 f(b)+\gamma_1 f'(a)+\delta_1 f'(b)=0 \\ \alpha_2 f(a)+\beta_2 f(b)+\gamma_2 f'(a)+\delta_2 f'(b)=0 \end{cases}$$

with real constants α_j, β_j, γ_j, and δ_j $(j=1,2)$. Then the eigenvalues of this problem form an infinite sequence $\lambda_1\le\lambda_2\le\lambda_3\le\ldots$ and $\lim_{n\to\infty}\lambda_n=+\infty$.

Proof: See Weinberger's *A First Course in Partial Differential Equations*, pp. 162-5.

Note: Chapter 11 of Strauss has the proof of analogous statements for symmetric boundary value problems associated with the Helmholtz equation $-\nabla^2 u=\lambda u$ on domains with "smooth" boundary in dimensions 2 and 3.

Let X_1, X_2, X_3, ... be eigenfunctions corresponding to the eigenvalues λ_1, λ_2, λ_3, ... of the problem in Theorem 1. Then according to Theorem 1 and the Gram-Schmidt orthogonalization process (cf. exercise #10 of Section 5.3), we may assume that $\langle X_i, X_j \rangle = \int_a^b X_i(x)\overline{X_j(x)}dx = 0$ if $i \neq j$. That is, the complete sequence of eigenfunctions $\{X_j\}_{j=1}^{\infty}$ is an orthogonal set of functions on (a,b).

Let f be any square-integrable function on (a,b). We can form the Fourier series $\sum_{n=1}^{\infty} c_n X_n(x)$ of f with respect to $\{X_j\}_{j=1}^{\infty}$, where $c_n = \dfrac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}$ for $n = 1,2,3,...$

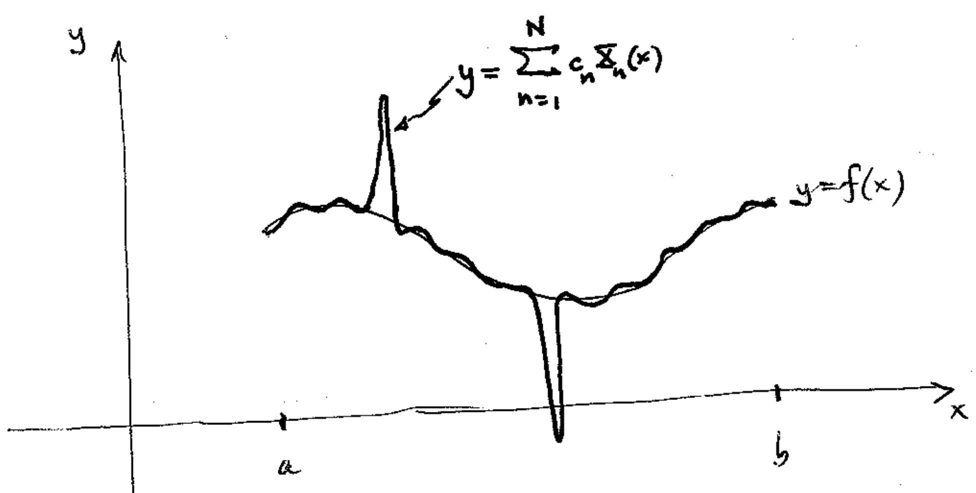
Question: In what sense does the Fourier series of f represent the function f on (a,b)?

There are three notions of convergence which can be used to help answer this question.

L^2**-convergence:** We say that the series $\sum_{n=1}^{\infty} c_n X_n(x)$ converges to f in the L^2 (or ***mean-square***) ***sense*** on (a,b) if

$$d_2\left(f, \sum_{n=1}^{N} c_n X_n\right) = \left(\int_a^b \left| f(x) - \sum_{n=1}^{N} c_n X_n(x) \right|^2 dx\right)^{1/2} \to 0 \text{ as } N \to \infty.$$

Note: In science, engineering, and communications, L^2-convergence is often called "convergence in energy" because $\left(\int_a^b |X(t)|^2 dt\right)^{1/2}$ represents the energy of the signal $X = X(t)$ on $a < t < b$.



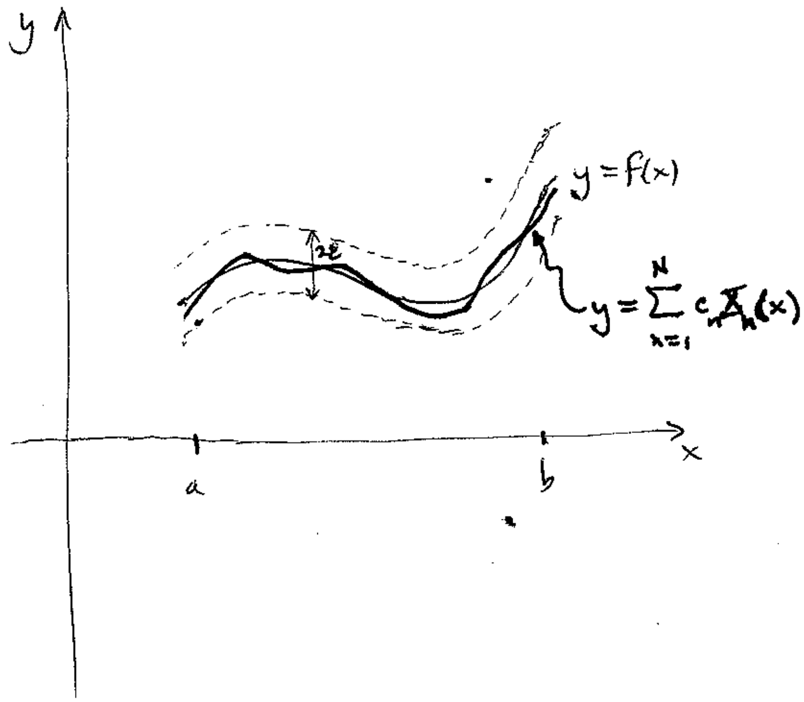
L^2-convergence: To each $\varepsilon > 0$ there corresponds a positive integer $N_0 = N_0(\varepsilon)$ such that

$$\left(\int_a^b \left| f(x) - \sum_{n=1}^{N} c_n X_n(x) \right|^2 dx\right)^{1/2} < \varepsilon$$

for all $N \geq N_0$.

Uniform Convergence: We say that the series $\sum_{n=1}^{\infty} c_n X_n(x)$ converges uniformly to f on $[a,b]$ provided

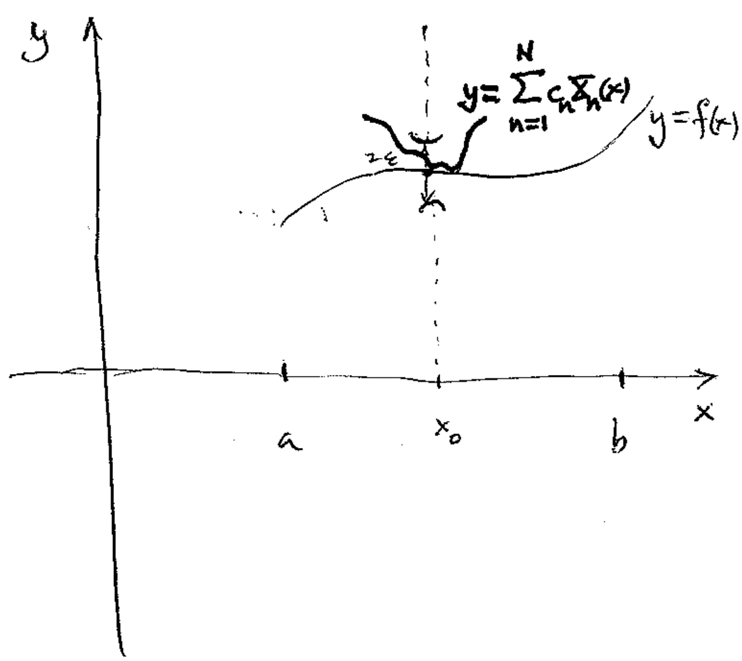
$$d_\infty\left(f, \sum_{n=1}^{N} c_n X_n\right) = \max_{a \le x \le b}\left| f(x) - \sum_{n=1}^{N} c_n X_n(x)\right| \to 0 \text{ as } N \to \infty.$$



Uniform Convergence: To each $\varepsilon > 0$ there corresponds an integer $N_0 = N_0(\varepsilon) \ge 1$ such that the graphs of the partial sums $\sum_{n=1}^{N} c_n X_n(x)$ for $N \ge N_0$ lie within the "tube" of radius ε centered on the graph of f over the entire interval $a \le x \le b$.

Pointwise Convergence: We say that the series $\sum_{n=1}^{\infty} c_n X_n(x)$ converges pointwise to f on (a,b) provided, for each point x_0 in (a,b),

$$\left| f(x_0) - \sum_{n=1}^{N} c_n X_n(x_0)\right| \to 0 \text{ as } N \to \infty.$$



Pointwise Convergence: To each x_0 in (a,b) and each $\varepsilon > 0$, there corresponds a positive integer $N_0 = N_0(\varepsilon, x_0)$ such that the graphs of the partial sums $\sum_{n=1}^{N} c_n X_n(x)$ for $N \ge N_0$ pass through the vertical "gate" of width 2ε centered at the point $(x_0, f(x_0))$ on the graph of f.

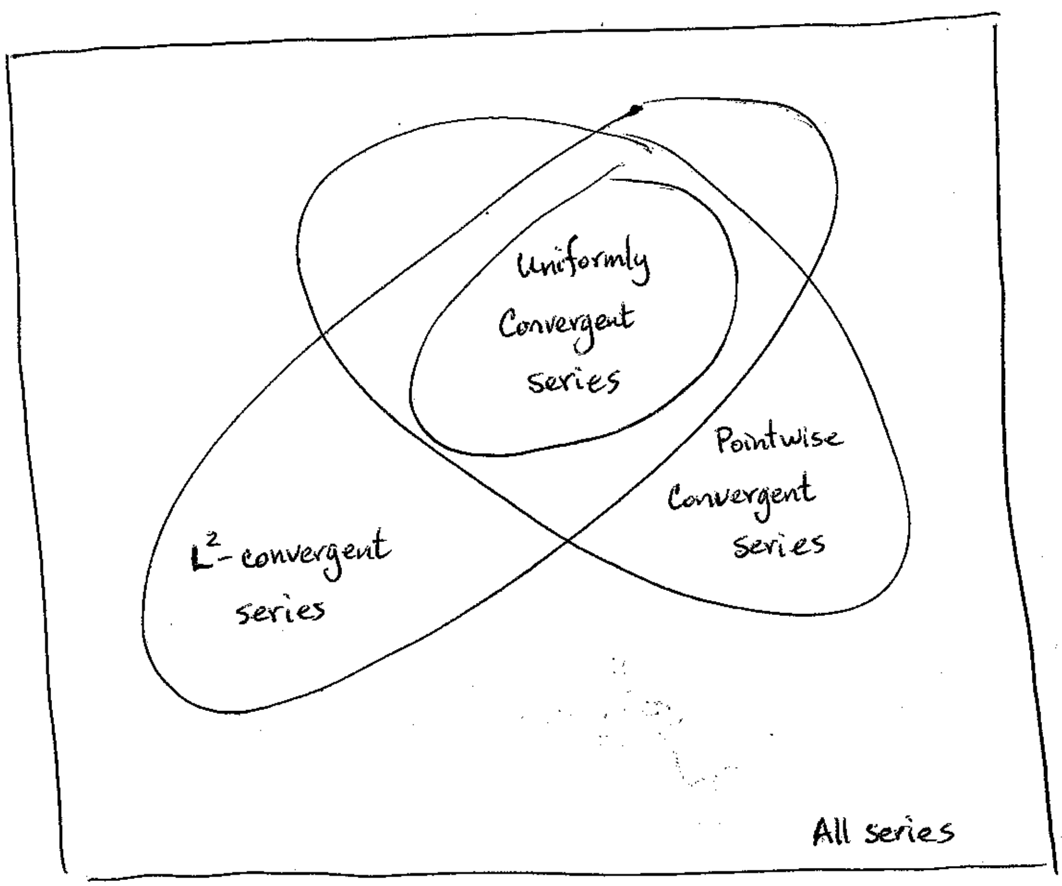
The following homework exercises in Section 5.4 relate the three types of convergence.

#2 $\begin{cases} \text{Uniform convergence implies pointwise convergence.} \\ \text{Uniform convergence implies } L^2-\text{convergence.} \end{cases}$

#3 $\begin{cases} \text{Pointwise convergence does not imply uniform convergence.} \\ \text{Pointwise convergence does not imply } L^2-\text{convergence.} \end{cases}$

#4 $\begin{cases} L^2-\text{convergence } \cancel{\text{convergence}} \text{ does not imply pointwise convergence.} \\ \text{(Therefore, by \#2 and \#4, } L^2-\text{convergence does not imply uniform convergence.)} \end{cases}$

Please see the solutions of these problems in the library for details. These relationships are illustrated by the following Venn diagram.



Convergence theorems for the three types of convergence of Fourier series are stated in the text and are reproduced on the next page. That page will be made available to you while you take hour exam III. We will not cover the proofs of these three theorems in this course; Mathematics 315 (Introduction to Real Analysis) covers proofs of results such as these. Instead, we will concentrate on learning how to use these theorems to help us solve partial differential equations problems.

Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b \text{ with any symmetric boundary conditions}$$

and let $\Phi=\{X_1,X_2,X_3,\dots\}$ be the complete orthogonal set of eigenfunctions for (1). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$f(x)\sim\sum_{n=1}^{\infty}A_nX_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle}\qquad (n=1,2,3,\dots).$$

Theorem 2. (Uniform Convergence) If
(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and
(ii) f satisfies the given symmetric boundary conditions,
then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 – Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)
(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.
(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty}A_nX_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

$(-l,l) = (-1,1)$ so $l=1$

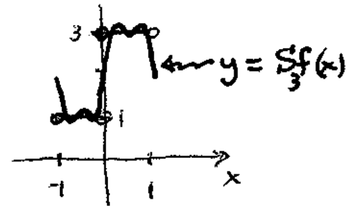
Example 1. (Similar to #7 in Section 5.4) Consider the 2-periodic function f defined on one period by

$$f(x)=\begin{cases}1 & \text{if } -1\le x<0,\\ 3 & \text{if } 0\le x<1.\end{cases} \qquad \Phi = \left\{1,\ \sin(n\pi x),\ \cos(n\pi x)\right\}_{n=1}^{\infty}$$

(a) Find the full Fourier series of f in the interval (-1,1).
(b) Write the sum of the first three nonzero terms of the full Fourier series of f and sketch the graph of this sum on the same axes as f.
(c) Does the full Fourier series of f converge in the L^2 – sense on (-1,1)?
(d) Does the full Fourier series of f converge pointwise on (-1,1)?
(e) Does the full Fourier series of f converge uniformly on [-1,1]?

(a) A routine calculation shows $f(x) \sim 2 + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi x)}{2k-1}$ on $(-1,1)$.

(b) $S_3 f(x) = 2 + \frac{4}{\pi}\left(\sin(\pi x) + \frac{1}{3}\sin(3\pi x)\right)$



(c) Since $\int_{-1}^{1} |f(x)|^2\,dx = 10 < \infty$, Theorem 3 shows that the full Fourier series of f converges to f in the L^2-sense: $\left(\int_{-1}^{1}\left|f(x) - 2 - \frac{4}{\pi}\sum_{k=1}^{N}\frac{\sin((2k-1)\pi x)}{2k-1}\right|^2 dx\right)^{1/2} \to 0$ as $N\to\infty$.

(d) Note that f and f' are piecewise continuous functions and f is 2-periodic on $(-\infty,\infty)$. Theorem 4^{∞} guarantees that the full Fourier series of f converges pointwise on $(-\infty,\infty)$ to $\frac{1}{2}\left[f(x^+)+f(x^-)\right]$. In particular, on $[-1,1]$ we have

$$2+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\pi x)}{2k-1} = \begin{cases} 1 & \text{if } -1<x<0,\\ 3 & \text{if } 0<x<1,\\ 2 & \text{if } x=-1, 0, \text{or } 1.\end{cases}$$

(e) The full Fourier series of f does <u>not</u> converge uniformly on $[-1,1]$. (Theorem 2 does <u>not</u> apply since f is not continuous on $[-1,1]$.) To see that $S_N f(x) \not\to f(x)$ uniformly on $[-1,1]$, note that the partial sums $2+\frac{4}{\pi}\sum_{k=1}^{N}\frac{\sin((2k-1)\pi x)}{2k-1}$ ($N=1,2,3,\ldots$) are continuous functions on $[-1,1]$ while their limit function $\frac{1}{2}\left[f(x^+)+f(x^-)\right]$ is discontinuous at $x=-1, 0,$ and 1. Then apply:

<u>Theorem</u> (Rudin, PMA, p. 150) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions which converges uniformly to the function f on an interval I, then f is continuous on I.

Example 2. Consider the 1-periodic function f given on one period by

$$\Phi = \{\sin(n\pi x)\}_{n=1}^{\infty} \qquad f(x)=\begin{cases}2x & \text{if } 0\le x<1/2,\\ 2-2x & \text{if } 1/2\le x\le 1.\end{cases} \Bigg\} \ (0,\ell) = (0,1) \text{ so } \ell = 1.$$

(a) Find the Fourier sine series of f in the interval [0,1].

(b) Write the sum of the first three nonzero terms of the Fourier sine series of f and sketch the graph of this sum on the same axes as f.

(c) Does the Fourier sine series of f converge in the L^2 – sense on [0,1]?

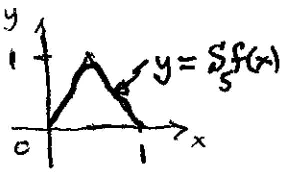
(d) Does the Fourier sine series of f converge pointwise on [0,1]?

(e) Does the Fourier sine series of f converge uniformly on [0,1]?

(a) In the lecture on Sec. 5.1 we computed $f(x) \sim \dfrac{8}{\pi^2}\displaystyle\sum_{k=0}^{\infty} \dfrac{(-1)^k \sin((2k+1)\pi x)}{(2k+1)^2}$ on $[0,1]$.

(b) In the lecture on Sec 5.1 we obtained $S_5 f(x) = \dfrac{8}{\pi^2}\left[\sin(\pi x) - \dfrac{\sin(3\pi x)}{9} + \dfrac{\sin(5\pi x)}{25}\right]$

and graphs of $S_5 f(x)$ and f:



(c) Since $\int_0^1 |f(x)|^2 dx = \frac{1}{3} < \infty$, Theorem 3 shows that the Fourier sine series of f converges to f in the L^2-sense; $\left(\int_0^1 \left| f(x) - \frac{8}{\pi^2}\sum_{k=0}^{N} \frac{(-1)^k \sin((2k+1)\pi x)}{(2k+1)^2}\right|^2 dx\right)^{1/2} \to 0$ as $N\to\infty$.

(d) Note that f is continuous on $[0,1]$ and f' is piecewise continuous on $[0,1]$. Theorem 4(i) shows that the Fourier sine series of f at x converges pointwise to $f(x)$ for each x in the open interval $(0,1)$. At the endpoints we have $f(0)=0$ and $f(1)=0$, and each term in the Fourier sine series of f is 0. Hence the Fourier sine series of f converges to 0 at $x=0$ and $x=1$. Thus, the Fourier sine series of f converges pointwise to f on $[0,1]$.

(e) Note that $f'(1/2)$ does not exist so Theorem 2 on uniform convergence does not apply. Instead, we use an ad hoc argument to show that the Fourier sine series of f does converge uniformly to f on $[0,1]$. Note that $f(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{k=0}^{\infty}\dfrac{(-1)^k\sin((2k+1)\pi x)}{(2k+1)^2}$ for all x in $[0,1]$ by part (d). Therefore

$$\max_{0\le x\le 1}\left| f(x) - \frac{8}{\pi^2}\sum_{k=0}^{N}\frac{(-1)^k \sin((2k+1)\pi x)}{(2k+1)^2}\right| = \max_{0\le x\le 1}\left|\frac{8}{\pi^2}\sum_{k=N+1}^{\infty}\frac{(-1)^k\sin((2k+1)\pi x)}{(2k+1)^2}\right|$$

$$\le \max_{0\le x\le 1}\frac{8}{\pi^2}\sum_{k=N+1}^{\infty}\left|\frac{(-1)^k\sin((2k+1)\pi x)}{(2k+1)^2}\right| \le \frac{8}{\pi^2}\sum_{k=N+1}^{\infty}\frac{1}{(2k+1)^2} < \frac{8}{\pi^2}\sum_{k=2N+3}^{\infty}\frac{1}{k^2} \to 0$$

as $N\to\infty$ because it is the "tail" of the convergent p-series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ $(p=2)$.

Least-Square Approximation and the L^2 – Theory of Fourier Series

Let $\{X_j\}_{j=1}^{\infty}$ be an orthogonal set of functions on (a,b) and let f be a square-integrable function on (a,b). Fix an integer $N \geq 1$.

Question: How should we choose the N constants $c_1, \ldots, c_N$ so that the square error

$$d_2\left(f, \sum_{n=1}^{N} c_n X_n\right) = \left(\int_a^b \left|f(x) - \sum_{n=1}^{N} c_n X_n(x)\right|^2 dx\right)^{1/2}$$

between f and the orthogonal polynomial $\sum_{n=1}^{N} c_n X_n$ is as small as possible?

Answer: (Theorem 5 in Section 5.4) Choose the constants to be the Fourier coefficients of f with respect to $\{X_j\}_{j=1}^{N}$ on (a,b):

$$c_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} \text{ for } n = 1, \ldots, N.$$

Note: The proof techniques used to answer the question above lead to two important results:

$$\text{Bessel's Inequality:} \quad \sum_{n=1}^{\infty} |c_n|^2 \int_a^b |X_n(x)|^2\, dx \leq \int_a^b |f(x)|^2\, dx,$$

which is valid for any orthogonal set of functions $\{X_j\}_{j=1}^{\infty}$ on (a,b), and

$$\text{Parseval's Identity:} \quad \sum_{n=1}^{\infty} |c_n|^2 \int_a^b |X_n(x)|^2\, dx = \int_a^b |f(x)|^2\, dx,$$

which is valid if $\{X_j\}_{j=1}^{\infty}$ is a **complete** orthogonal set of functions on (a,b). (See p. 128 for the definition of complete.) It is important to point out that the coefficients c_n in both Bessel's inequality and Parseval's identity are the Fourier coefficients of f with respect to $\{X_j\}_{j=1}^{\infty}$.

Example 3. (#14 in Section 5.4) Find the sum $\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6}$.

Solution: It is routine to check that the Fourier sine series of $f(x) = x^3 - x$ on $[0,1]$ is $\frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^3}$. By Parseval's identity we have

$$\sum_{n=1}^{\infty} \left|\frac{12(-1)^n}{\pi^3 n^3}\right|^2 \int_0^1 \sin^2(n\pi x)dx = \int_0^1 \left(x^3 - x\right)^2 dx.$$

Evaluating the integrals gives $\sum_{n=1}^{\infty} \left|\frac{12(-1)^n}{\pi^3 n^3}\right|^2 \frac{1}{2} = \frac{8}{105}$ and hence,

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{8}{105} \cdot \frac{2\pi^6}{144} = \frac{\pi^6}{945}.$$