Mathematics 5325
Homework 12

Due Date: ____________________

Name: _________________________

Solve ONE of the following two problems. CIRCLE the letter of the problem you want me to grade.

A. Use Fourier transform methods to find a formula for the solution to

$$u_{xx} + u_{yy} = f(x,y) \quad \text{for} \quad -\infty < x < \infty,\ 0 < y < \infty,$$

which satisfies

$$u(x,0) = \varphi(x) \ \text{ and } \ \lim_{y\to\infty} u(x,y) = 0 \quad \text{for each } x \text{ in } (-\infty,\infty).$$

B. Use Fourier transform methods to find a formula for the solution to

$$u_{tt} - u_{xx} = f(x,t) \quad \text{for} \quad -\infty < x < \infty,\ 0 < t < \infty,$$

which satisfies

$$u(x,0) = \phi(x) \ \text{ and } \ u_t(\mathrm{x},0) = \psi(\mathrm{x}) \ \text{ if } \ -\infty < x < \infty.$$

(B) $$u_{tt} - u_{xx} \overset{①}{=} f(x,t) \quad \text{if} \quad -\infty < x < \infty,\ 0 < t < \infty,$$

$$u(x,0) \overset{②}{=} \varphi(x) \quad \text{and} \quad u_t(x,0) \overset{③}{=} \psi(x) \quad \text{if} \quad -\infty < x < \infty.$$

Let $u = u(x,t)$ solve ①-②-③. Then taking the Fourier transform with respect to x, holding t fixed, of $u_{tt}(x,t) - u_{xx}(x,t) = f(x,t)$, we find

$$\mathcal{F}\big(u_{tt}(\cdot,t) - u_{xx}(\cdot,t)\big)(\xi) = \mathcal{F}\big(f(\cdot,t)\big)(\xi).$$

Using the linearity property (property ①) and the transform of a derivative property (property ②) of Fourier transforms gives

$$\mathcal{F}\big(u_{tt}(\cdot,t)\big)(\xi) - (i\xi)^2\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \hat{f}(\xi,t).$$

Interchanging the Fourier transform (an improper integral with respect to x) and differentiation with respect to t (twice) yields

$$(*) \qquad \frac{\partial^2}{\partial t^2}\mathcal{F}\big(u(\cdot,t)\big)(\xi) + \xi^2\mathcal{F}\big(u(\cdot,t)\big)(\xi) = \hat{f}(\xi,t).$$

This is a second order, linear, nonhomogeneous ODE in the variable t, with parameter ξ, so

$$\mathcal{F}\big(u(\cdot,t)\big)(\xi) = U_h(\xi,t) + U_p(\xi,t)$$

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where U_h is the general solution of the homogeneous equation corresponding to (*) and U_p is a particular solution of (*). To solve

$$\frac{\partial^2 U}{\partial t^2} + \xi^2 U = 0$$

we assume $U = e^{rt}$. This leads to $r^2 e^{rt} + \xi^2 e^{rt} = 0$ so $r^2 + \xi^2 = 0$ and hence $r = \pm i\xi$. Therefore

$$U_h(\xi,t) = c_1(\xi)e^{i\xi t} + c_2(\xi)e^{-i\xi t}$$

where c_1 and c_2 are arbitrary functions of a single real variable ξ.

To find a particular solution of (*), we use variation of parameters:

$$U_p(\xi,t) = V_1(\xi,t)e^{i\xi t} + V_2(\xi,t)e^{-i\xi t}$$

where the Wronskian of $e^{i\xi t}$ and $e^{-i\xi t}$ is

$$W(e^{i\xi t}, e^{-i\xi t}) = \begin{vmatrix} e^{i\xi t} & e^{-i\xi t} \\ i\xi e^{i\xi t} & -i\xi e^{-i\xi t} \end{vmatrix} = -2i\xi,$$

and

$$V_1(\xi,t) = \int \frac{-\hat{f}(\xi,t)e^{-i\xi t}}{W(e^{i\xi t}, e^{-i\xi t})}\,dt = \int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau$$

$$V_2(\xi,t) = \int \frac{\hat{f}(\xi,t)e^{i\xi t}}{W(e^{i\xi t}, e^{-i\xi t})}\,dt = \int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{-2i\xi}\,d\tau.$$

Therefore

$$U_p(\xi,t) = e^{i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau + e^{-i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{-2i\xi}\,d\tau$$

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and hence, the general solution of (*) is

$$\mathcal{F}(u(\cdot,t))(\xi) = c_1(\xi)e^{i\xi t} + c_2(\xi)e^{-i\xi t} + e^{i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau + e^{-i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{-2i\xi}\,d\tau$$

We need to apply the initial conditions ② and ③ to identify $c_1(\xi)$ and $c_2(\xi)$:

$$(**) \qquad \hat{\varphi}(\xi) = \mathcal{F}(u(\cdot,t))(\xi)\Big|_{t=0} = c_1(\xi) + c_2(\xi).$$

$$\hat{\psi}(\xi) = \mathcal{F}(u_t(\cdot,t))(\xi)\Big|_{t=0} = \frac{\partial}{\partial t}\mathcal{F}(u(\cdot,t))(\xi)\Big|_{t=0}$$

$$= \left(i\xi c_1(\xi)e^{i\xi t} - i\xi c_2(\xi)e^{-i\xi t} + i\xi e^{i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{-i\xi\tau}}{2i\xi}\,d\tau + e^{i\xi t}\frac{\hat{f}(\xi,t)e^{-i\xi t}}{2i\xi} - i\xi e^{-i\xi t}\int_0^t \frac{\hat{f}(\xi,\tau)e^{i\xi\tau}}{-2i\xi}\,d\tau \right.$$

$$\left. + e^{-i\xi t}\frac{\hat{f}(\xi,t)e^{i\xi t}}{-2i\xi} \right)\Bigg|_{t=0}$$

$$(***) \qquad \hat{\psi}(\xi) = i\xi c_1(\xi) - i\xi c_2(\xi) + \cancel{\frac{\hat{f}(\xi,0)}{2i\xi}} - \cancel{\frac{\hat{f}(\xi,0)}{2i\xi}}.$$

Multiplying (**) by $i\xi$ and adding the result to (***) yields

$$\hat{\psi}(\xi) + i\xi\hat{\varphi}(\xi) = 2i\xi c_1(\xi) \qquad \text{so} \qquad c_1(\xi) = \frac{1}{2}\hat{\varphi}(\xi) + \frac{1}{2}\frac{\hat{\psi}(\xi)}{i\xi}.$$

Subtracting (***) from $i\xi$ times (**) yields

$$-\hat{\psi}(\xi) + i\xi\hat{\varphi}(\xi) = 2i\xi c_2(\xi) \qquad \text{so} \qquad c_2(\xi) = \frac{1}{2}\hat{\varphi}(\xi) - \frac{1}{2}\frac{\hat{\psi}(\xi)}{i\xi}.$$

Therefore

$$(****) \qquad \mathcal{F}(u(\cdot,t))(\xi) = \frac{1}{2}\hat{\varphi}(\xi)e^{i\xi t} + \frac{1}{2}\frac{\hat{\psi}(\xi)}{i\xi}e^{i\xi t} + \frac{1}{2}\hat{\varphi}(\xi)e^{-i\xi t} - \frac{1}{2}\cdot\frac{\hat{\psi}(\xi)}{i\xi}e^{-i\xi t}$$

$$+ \frac{1}{2}\int_0^t \frac{\hat{f}(\xi,\tau)}{i\xi}e^{-i\xi(\tau-t)}\,d\tau - \frac{1}{2}\int_0^t \frac{\hat{f}(\xi,\tau)}{i\xi}e^{i\xi(\tau-t)}\,d\tau.$$

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To proceed further, we need the following two Fourier transform facts:

(a) $\widehat{g(\cdot - a)}(\xi) = e^{-ia\xi}\,\hat{g}(\xi)$ for every real number a and every absolutely integrable function g on $(-\infty, \infty)$;

(b) if $G(x) = \int_0^x g(s)\,ds$ then $\hat{G}(\xi) = \dfrac{\hat{g}(\xi)}{i\xi}$.

Fact (a) is #4 on the handout "Exercises for Fourier Transforms". Fact (b) follows easily from the identity for the Fourier transform of a derivative: $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$.

Applying (a) we have $\hat{\varphi}(\xi)e^{i\xi t} = \widehat{\varphi(\cdot + t)}(\xi)$ and $\hat{\varphi}(\xi)e^{-i\xi t} = \widehat{\varphi(\cdot - t)}(\xi)$.

Applying (b) and (a) we have:

$$\frac{\hat{\psi}(\xi)}{i\xi}e^{i\xi t} = \hat{\Psi}(\xi)e^{i\xi t} = \widehat{\Psi(\cdot + t)}(\xi) \qquad \left(\text{where } \Psi(x) = \int_0^x \psi(s)\,ds\right),$$

$$\frac{\hat{\psi}(\xi)}{i\xi}e^{-i\xi t} = \hat{\Psi}(\xi)e^{-i\xi t} = \widehat{\Psi(\cdot - t)}(\xi),$$

$$\frac{\hat{f}(\xi,\tau)}{i\xi}e^{-i\xi(\tau - t)} = \hat{F}(\xi,\tau)e^{i\xi(t-\tau)} = \widehat{F(\cdot + t - \tau, \tau)}(\xi) \qquad \left(\text{where } F(x,\tau) = \int_0^x f(s,\tau)\,ds\right)$$

$$\frac{\hat{f}(\xi,\tau)}{i\xi}e^{i\xi(\tau - t)} = \hat{F}(\xi,\tau)e^{i\xi(\tau-t)} = \widehat{F(\cdot + \tau - t, \tau)}(\xi).$$

Substituting these expressions into (****) gives

$$\mathcal{F}(u(\cdot,t))(\xi) = \tfrac{1}{2}\mathcal{F}\big(\varphi(\cdot+t)\big)(\xi) + \tfrac{1}{2}\mathcal{F}\big(\Psi(\cdot+t)\big)(\xi) + \tfrac{1}{2}\mathcal{F}\big(\varphi(\cdot-t)\big)(\xi) - \tfrac{1}{2}\mathcal{F}\big(\Psi(\cdot-t)\big)(\xi)$$
$$+\frac{1}{2}\int_0^t \mathcal{F}\big(F(\cdot+t-\tau,\tau)\big)(\xi)\,d\tau - \frac{1}{2}\int_0^t \mathcal{F}\big(F(\cdot+\tau-t,\tau)\big)(\xi)\,d\tau.$$

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Interchanging the order of integration in the last two terms and then using linearity of the Fourier transform leads to

$$\mathcal{F}(u(\cdot,t))(\xi) = \mathcal{F}\left(\tfrac{1}{2}\varphi(\cdot+t)+\tfrac{1}{2}\varphi(\cdot-t)+\tfrac{1}{2}\Psi(\cdot+t)-\tfrac{1}{2}\Psi(\cdot-t)\right)(\xi)$$

$$+\ \mathcal{F}\left(\tfrac{1}{2}\int_0^t F(\cdot+t-\tau,\tau)\,d\tau\right)(\xi) - \mathcal{F}\left(\tfrac{1}{2}\int_0^t F(\cdot+\tau-t,\tau)\,d\tau\right)(\xi)$$

$$= \mathcal{F}\left(\tfrac{1}{2}\left[\varphi(\cdot+t)+\varphi(\cdot-t)\right]+\tfrac{1}{2}\left[\Psi(\cdot+t)-\Psi(\cdot-t)\right]+\tfrac{1}{2}\int_0^t\left[F(\cdot+t-\tau,\tau)-F(\cdot+\tau-t,\tau)\right]d\tau\right)(\xi)$$

The inversion theorem then yields

$$u(x,t)=\tfrac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\tfrac{1}{2}\left[\int_0^{x+t}\psi(s)\,ds-\int_0^{x-t}\psi(s)\,ds\right]+\tfrac{1}{2}\int_0^t\left[\int_0^{x+t-\tau}f(s,\tau)\,ds-\int_0^{x+\tau-t}f(s,\tau)\,ds\right]d\tau$$

or equivalently

$$\boxed{u(x,t)=\tfrac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\tfrac{1}{2}\int_{x-t}^{x+t}\psi(s)\,ds+\tfrac{1}{2}\int_0^t\int_{x+\tau-t}^{x+t-\tau}f(s,\tau)\,ds\,d\tau}$$

for all $-\infty<x<\infty$ and $0<t<\infty$.