1. (25 pts.) Find all second-degree polynomial functions of two real variables,

\[ u(x,t) = ax^2 + bxt + ct^2 + dx + et + f \]

where \( a, b, c, d, e, \) and \( f \) are real constants, which are solutions in the \( xt \)-plane of the one-dimensional diffusion equation

\[ u_t - ku_{xx} = 0. \]

\[ \begin{align*}
  u_t &= bx + 2ct + e \\
  u_x &= 2ax + bt + d \\
  u_{xx} &= 2a
\end{align*} \]

We want \( u_t = ku_{xx} \) so substituting gives

\[ bx + 2ct + e = k2a. \]

This is to hold for all \(-\infty < x < \infty\) and \(-\infty < t < \infty\).

Therefore we must have "like" coefficients equal on the left and right sides of this identity. That is, \( b = 0, 2c = 0, \) and \( e = 2ka. \) Thus

\[ u(x,t) = ax^2 + dx + 2kat + f \]

where \( a, d, \) and \( f \) are arbitrary real constants.

\[ u(x,t) = a(x^2 + 2kt) + dx + f \]
2. (25 pts.) Find the general solution of

\[ yu_x - xu_y = 0 \]

in the \( xy \)-plane. Sketch several characteristic curves of this partial differential equation.

The solution \( u = u(x,y) \) is constant along all curves in the \( xy \)-plane which satisfy the characteristic equation of this pde:

\[ \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = \frac{-x}{y} \]

Separating variables and integrating yields

\[ \frac{y^2}{2} = \int y \, dy = \int -x \, dx = -\frac{x^2}{2} + c_1, \]

\[ \Rightarrow x^2 + y^2 = c \quad \text{(where } c = 2c_1, \text{)} \]

Along each such curve we have

\[ u(x,y) = u(x, \pm \sqrt{c-x^2}) = u(0, \pm \sqrt{c}) = f(c). \]

Therefore \( u(x,y) = f(x^2 + y^2) \) where \( f \) is an arbitrary differentiable function of a single real variable. 20 pts. to here.

The characteristic curves \( x^2 + y^2 = c \)

of the pde are circles centered at the origin. 2.5 pts. to here.
3. (25 pts.) Consider the linearized gas dynamics equations

\[
\frac{\partial \mathbf{v}}{\partial t} + \frac{c_0^2}{\rho_0} \text{grad}(\rho) = 0
\]

\[
\frac{\partial \rho}{\partial t} + \rho_0 \text{div}(\mathbf{v}) = 0
\]

where \( \rho_0 \) is the density and \( c_0 \) is the speed of sound in still air. Verify that if \( \text{curl}(\mathbf{v}) = 0 \) when \( t = 0 \), then \( \text{curl}(\mathbf{v}) = 0 \) at all later times.

Suppose \( \mathbf{v} = \mathbf{v}(x, y, z, t) \) and \( \rho = \rho(t) \) are solutions to the gas dynamics equation. Consider the function

\[
\mathbf{\tilde{w}} = \text{curl}(\mathbf{v}(x, y, z, t)).
\]

Then

\[
\frac{\partial \mathbf{\tilde{w}}}{\partial t} = \frac{\partial}{\partial t} \text{curl}(\mathbf{v}(x, y, z, t)) = \text{curl} \left( \frac{\partial \mathbf{v}}{\partial t} \right)
\]

because \( \mathbf{v} = \mathbf{v}(x, y, z, t) \). This is a \( C^2 \)-function so

\[
\frac{\partial}{\partial t} \left( \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial z} \right) = \frac{\partial}{\partial y} \left( \frac{\partial v_1}{\partial t} \right) - \frac{\partial}{\partial z} \left( \frac{\partial v_2}{\partial t} \right),
\]

etc. But then

\[
\frac{\partial \mathbf{\tilde{w}}}{\partial t} = \text{curl} \left( - \frac{c_0^2}{\rho_0} \text{grad}(\rho) \right) = -\frac{c_0^2}{\rho_0} \text{curl}(\text{grad}(\rho)) = \mathbf{0}
\]

from the first gas dynamics equation and the fact that the curl of a gradient of a \( C^2 \)-function is zero. It follows that the function \( \mathbf{\tilde{w}} \) is independent of \( t \); i.e., for all \( t \geq 0 \) and all \( (x, y, z) \in \mathbb{R}^3 \),

\[
\text{curl}(\mathbf{v}(x, y, z, t)) = \mathbf{\tilde{w}}(x, y, z, t) = \mathbf{\tilde{w}}(x, y, z, 0) = \text{curl}(\mathbf{v}(x, y, z, 0)) = \mathbf{0}.
\]
4. (25 pts.) Consider the partial differential equation

\( u_{xx} - 3u_{tt} - 4u_{tt} = 0. \)

(a) Classify (*) as elliptic, hyperbolic, or parabolic.
(b) Find the general solution of (*) in the \( xt \)-plane.
(c) If \( \phi \) and \( \psi \) are \( C^2 \) and \( C^1 \) functions of a single real variable, respectively, find the solution of (*) which satisfies the initial conditions

\[ u(x, 0) = \phi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x) \quad \text{for all } -\infty < x < \infty. \]

5 pts.

(a) \( B^2 - 4AC = (-3)^2 - 4(1)(-4) = 9 + 16 = 25 > 0. \) \( (*) \) is hyperbolic.

(b) \( (*) \) is equivalent to \( \left( \frac{\partial^2}{\partial x^2} - 3 \frac{\partial}{\partial x} + \frac{4}{4} \frac{\partial^2}{\partial t^2} \right) u = 0 \) and thus to

\[ \left( \frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) u = 0. \]

The change of coordinates \( \left\{ \begin{array}{l} z = 3x - 4t = (-4x - t) \\
\eta = 5x - 4t = x - t \end{array} \right. \)

transforms \( (*) \) into \( \frac{\partial^2 u}{\partial \eta \partial \eta} = 0 \) so the general solution of \( (*) \) is

\[ u = f(z) + g(\eta) = f(4x + t) + g(x - t) \]

where \( f \) and \( g \) are arbitrary twice-differentiable functions of a single real variable.

(c) Using (b) and the initial conditions in (c), we have

\( \phi(x) = u(x, 0) = f(4x) + g(x) \quad \text{(-\infty < x < \infty)} \)

\( \psi(x) = u_t(x, 0) = f(4x) - g'(x) \quad \text{(-\infty < x < \infty)} \)

Differentiating (1) yields

\( \phi'(x) = 4f'(4x) + g'(x) \quad \text{(-\infty < x < \infty)} \).

Adding (2) and (1') produces

\( \phi'(x) + \psi(x) = 5f'(4x) \quad \text{(-\infty < x < \infty)} \)

or equivalently

\( f'(z) = \frac{1}{5} \phi'(\frac{z}{4}) + \frac{1}{5} \psi(\frac{z}{4}) \quad \text{(-\infty < z < \infty)} \).

Integrating yields (over)
\[
(5) \quad f(z) = \frac{4}{5} \varphi \left( \frac{z}{4} \right) + \frac{4}{5} \int_0^{z/4} \psi(s) \, ds + c, \quad (-\infty < z < \infty).
\]

Therefore, using (5) and (1) we have

\[
(6) \quad g(x) = \varphi(x) - f(4x) = \varphi(x) - \left( \frac{4}{5} \varphi(x) + \frac{4}{5} \int_0^{x/4} \psi(s) \, ds + c_1 \right) = \frac{1}{5} \varphi(x) - \frac{4}{5} \int_0^x \psi(s) \, ds - c_1, \quad (-\infty < x < \infty).
\]

It follows from (6), (5), and (6) that

\[
u(x, t) = f(4x + t) + g(x - t)
\]

\[
= \frac{4}{5} \varphi \left( \frac{4x + t}{4} \right) + \frac{4}{5} \int_0^{(x + t)/4} \psi(s) \, ds + \gamma + \frac{1}{5} \varphi(x - t) - \frac{4}{5} \int_0^{x - t} \psi(s) \, ds - \gamma
\]

\[
u(x, t) = \frac{1}{5} \left[ 4 \varphi \left( x + \frac{t}{4} \right) + \varphi(x - t) \right] + \frac{4}{5} \int_0^{x + \frac{t}{4}} \psi(s) \, ds - \frac{1}{5} \int_0^{x - t} \psi(s) \, ds
\]
Bonus. (25 pts.) A homogeneous solid material occupying \( D = \{ (x, y, z) \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 100 \} \) is completely insulated and its initial temperature at position \((x, y, z)\) in \( D \) is \( \frac{200}{\sqrt{x^2 + y^2 + z^2}} \).

(a) Write (without proof or derivation) the partial differential equation and initial/boundary conditions that completely govern the temperature \( u(x, y, z, t) \) at position \((x, y, z)\) in \( D \) and time \( t \geq 0 \).

(b) Use Gauss' divergence theorem to help show that the heat energy \( H(t) = \iiint_D c \rho u(x, y, z, t) \, dV \) of the material in \( D \) at time \( t \) is a constant function of time. Here \( c \) and \( \rho \) denote the (constant) specific heat and mass density, respectively, of the material in \( D \).

(c) Compute the (constant) steady-state temperature that the material in \( D \) reaches after a long time.

\[ a) \begin{aligned}
    \begin{cases}
        u_t - k (u_{xx} + u_{yy} + u_{zz}) & = 0 & \text{if} & & 4 < x^2 + y^2 + z^2 < 100 \text{ and } t > 0 \\
        \nabla u \cdot \vec{n} & = 0 & \text{if} & & x^2 + y^2 + z^2 = 4 \text{ or } x^2 + y^2 + z^2 = 100 \text{ and } t > 0 \\
        u(x, y, z, 0) & = \frac{200}{\sqrt{x^2 + y^2 + z^2}} & \text{if} & & 4 \leq x^2 + y^2 + z^2 \leq 100
    \end{cases}
\end{aligned} \]

\[ b) \begin{aligned}
    H'(t) & = \frac{d}{dt} \iiint_D c \rho u(x, y, z, t) \, dV \\
    & = \iiint_D \frac{\partial}{\partial t} (c \rho u) \, dV \\
    & = \iiint_D \nabla \cdot (c \rho \nabla u) \, dV \\
    & = \iiint_D \nabla \cdot (c \rho \nabla u) \, dV \\
    & = 0
\end{aligned} \]

Therefore, the heat energy \( H = H(t) \) is a constant function of \( t \).

\[ c) \begin{aligned}
    \text{Let } U = \lim_{t \to \infty} u(x, y, z, t) \text{ be the constant steady-state temperature that the material in } D \text{ reaches after a long time. From part (b), } H(0) = H(t) \text{ for all } t > 0. \text{ Therefore}
\end{aligned} \]

\[ H(0) = \lim_{t \to 0} H(t) = \lim_{t \to \infty} \iiint_D c \rho u(x, y, z, t) \, dV = \iiint_D \lim_{t \to \infty} u(x, y, z, t) \, dV \]

\[ = \iiint_D c \rho \, dV = c \rho U \text{ vol}(D). \quad \text{(OVER)} \]
Solving for \( U \) we find

\[
U = \frac{H(0)}{c_p \text{vol}(D)}
\]

But computations yield

\[
\text{vol}(D) = \frac{4}{3} \pi r_1^3 - \frac{4}{3} \pi r_2^3 = \frac{4}{3} \pi \left( 10^3 - 2^3 \right) = \frac{3968 \pi}{3}
\]

and

\[
H(0) = \iiint_D cp(x, y, z, \theta) dV = cp \iiint_D \frac{200}{\sqrt{x^2 + y^2 + z^2}} dV
\]

\[
= cp \iiint_D \int_0^{\pi} \int_0^{\frac{10}{r}} \int_0^{\pi} r^2 \sin \theta dr d\theta d\phi
\]

\[
= cp \left( 2\pi \right) (-\cos \phi) \bigg|_0^{\frac{10}{r}} \left( 100r^2 \right) \bigg|_2^0
\]

\[
= 38,400 \pi c_p
\]

Therefore

\[
U = \frac{38,400 \pi c_p}{cp \left( \frac{3968 \pi}{3} \right)} = \frac{900}{31} \approx 29.0
\]
Math 325  
Exam I  
Summer 2009  

$n = 15$  
standard deviation: 20.7  
mean: 72.6  

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