1. (25 pts.) (a) Find the characteristic curves of $yu_x - xu_y = 0$ in the $xy$-plane.
(b) Sketch and identify two characteristic curves of this partial differential equation.
(c) Write the general solution of this partial differential equation in the $xy$-plane.
(d) Find the solution of this partial differential equation which satisfies $u(x, 0) = x^6$ for all real $x$.
(e) In what region of the $xy$-plane is the solution in part (d) uniquely determined?

6 pts. (a) The characteristic curves of the p.d.e. satisfy
\[ \frac{dy}{dx} = \frac{b(x, y)}{a(x, y)} = \frac{-x}{y}. \]
Separating variables gives $y\,dy = -x\,dx$ and integrating both sides yields
\[ \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1, \text{ or equivalently } \frac{x^2 + y^2}{c} = C. \] ($C = 2c_1$ is an arbitrary constant)

3 pts. (b) The characteristic curves are circles.

8 pts. (c) The solutions to the p.d.e. are constant along characteristic curves. Along such a curve $x^2 + y^2 = C$ we have
\[ u(x, y) = u(x, \pm \sqrt{c-x^2}) = u(0, \pm \sqrt{c}) = f(c). \]
Therefore the general solution to the p.d.e. in the $xy$-plane is
\[ u(x, y) = f(x^2 + y^2) \]
where $f$ is any $C^1$-function of a single real variable.

4 pts. (d) $x^6 = u(x, 0) = f(x^2 + 0^2) = f(x^2)$ for all real $x$. Thus $f(t) = t^3$ for every $t \geq 0$, so $u(x, y) = f(x^2 + y^2) = (x^2 + y^2)^3$ for all $(x, y)$ in the plane.

2 pts. (e) The solution in part (d) is uniquely determined in the entire $xy$-plane.
2. (25 pts.) An elastic string is held fixed at the endpoints and plucked. Assuming that air exerts a resistance at each point of the string which is proportional to the string’s velocity at that point, give a careful derivation of the partial differential equation that governs the small vibrations of the string.

\[ u = u(x, t) \]

Fix \( t > 0 \) and suppose the profile of the string at this instant is as indicated in the diagram. Consider the segment of string between \( x \) and \( x + \Delta x \). The contact forces \( \vec{T}_1 \) and \( \vec{T}_2 \) that act at the endpoints of the segment are tangential to the string at the respective points. If \( \alpha \) and \( \beta \) are the angles these contact forces make with respect to the horizontal, then

\[ \tan(\alpha) = \text{slope of the tangent line to the profile at } x = u_x(x, t) \]

\[ \sin(\alpha) = \frac{u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}} \quad \text{and} \quad \cos(\alpha) = \frac{1}{\sqrt{1 + u_x^2(x, t)}} \]

Similarly,

\[ \sin(\beta) = \frac{u_x(x + \Delta x, t)}{\sqrt{1 + u_x^2(x + \Delta x, t)}} \quad \text{and} \quad \cos(\beta) = \frac{1}{\sqrt{1 + u_x^2(x + \Delta x, t)}} \]

Applying Newton's second law, \( \mathbf{ma} = \vec{F}_{\text{net}} \), to the system of particles comprising the segment of string from \( x \) to \( x + \Delta x \), we have upon resolving into components:

\[ (\text{Horizontal}) \quad |\vec{T}_2| \cos(\beta) - |\vec{T}_1| \cos(\alpha) = 0 \]

\[ (\text{Vertical}) \quad |\vec{T}_2| \sin(\beta) - |\vec{T}_1| \sin(\alpha) - \int_x^{x+\Delta x} \rho u_x(\vec{x}, t) \, dx = \int_x^{x+\Delta x} \rho(\vec{x}) u_t(\vec{x}, t) \, dx \]

(\text{OVER})
where $r$ is a positive constant and $\rho(x)$ is the linear density of the string at $x$ in the interval $[x, x+ax]$. (Here we have made use of the assumption that the air exerts a resistive force at each point that is proportional to the string's velocity $u_x$ at that point.) Equations (3), (5), and (6) imply

\[ |\vec{T}_2| = |\vec{T}_1| \sqrt{\frac{1 + u_x^2(x+ax,t)}{1 + u_x^2(x,t)}} \]

so substituting from (8) into (7) and using (2) and (4) yields

\[ \frac{1}{\sqrt{1 + u_x^2(x,t)}} \left( u_x(x+ax,t) - u_x(x,t) \right) dx + \int_{x}^{x+ax} r u_t(x,t) dx = \int_{x}^{x+ax} \rho(x) u_t(x,t) dx. \]

Dividing (9) by $ax$ and letting $ax \to 0$ produces

\[ \frac{|\vec{T}|}{\sqrt{1 + u_x^2(x,t)}} \left( u_{xx}(x,t) - r u_t(x,t) = \rho(x) u_{tt}(x,t) \right). \]

For small vibrations, $u(x,t)$ and $u_x(x,t)$ are quite small so $\sqrt{1 + u_x^2(x,t)} \approx 1$ for all $0 \leq x \leq L$. Thus (8) implies the magnitude of the contact force is approximately constant; say $|\vec{T}(x,u(x,t),u_x(x,t))| = \text{constant} = T_0$. Thus (10) becomes

\[ \frac{T_0}{\sqrt{1 + u_x^2(x,t)}} \left( u_{xx}(x,t) - r u_t(x,t) = \rho(x) u_{tt}(x,t) \right). \]

If the string is homogeneous, say $\rho(x) = \text{constant} = \rho_0$, and we use the small vibrations approximation, $\sqrt{1 + u_x^2(x,t)} \approx 1$ for all $0 \leq x \leq L$, then (11) becomes

\[ T_0 u_{xx}(x,t) - r u_t(x,t) = \rho_0 u_{tt}(x,t) \]

where $T_0$, $r$, and $\rho_0$ are positive constants.
3. (25 pts.) Classify the following second order partial differential equations as hyperbolic, parabolic, elliptic, or nonlinear. Find the general \( C^2 \) solution in the plane whenever possible.

(a) \( 9u_{xx} + 6u_{xy} + u_{yy} + 100u = 0 \)

(b) \( u_{xx} + u_{yy} + u_z = 0 \)

(c) \( u_{xx} - 2u_{xy} + u_{yy} - 3u_{xx} + 2u_x - 2u_y = 0 \)

\( \Rightarrow u_{xx} - 4u_{xy} + 3u_{yy} + 2u_x - 2u_y = 0 \).

(a) \( B^2 - 4AC = 6^2 - 4(9)(1) = 0 \) so the p.d.e. is \( \text{parabolic} \).

(b) The p.d.e. is \( \text{nonlinear} \) because of the term \( u_{xx} \).

(c) \( B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0 \) so the p.d.e. is \( \text{hyperbolic} \).

8 pts. To solve the p.d.e. in (a) note that it is equivalent to 

\[ (3\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(3\frac{\partial}{\partial x} + \frac{\partial}{\partial y})u + 100u = 0 \]

Let \( \{ \begin{align*}
\xi &= 3x + y \\
\eta &= x - 3y
\end{align*} \)

Then by the chain rule, as operators we have

\[ \frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial\eta}{\partial x} \frac{\partial}{\partial \eta} = 3 \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \]

and

\[ \frac{\partial}{\partial y} = \frac{\partial\xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial\eta}{\partial y} \frac{\partial}{\partial \eta} = 3 \frac{\partial}{\partial \xi} - 3 \frac{\partial}{\partial \eta} \]

Therefore

\[ 3 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} = 3(3 \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}) + (\frac{\partial}{\partial \xi} - 3 \frac{\partial}{\partial \eta}) = 10 \frac{\partial}{\partial \xi} \]

Thus the p.d.e.

12 pts. To solve the p.d.e. in (a) is equivalent to 

\[ (10 \frac{\partial}{\partial \xi})(10 \frac{\partial}{\partial \eta})u + 100u = 0 \]

\[ \Rightarrow \frac{\partial^2 u}{\partial \xi^2} + u = 0 \]

Consequently, \( u = c_1(\eta) \cos(\xi) + c_2(\eta) \sin(\xi) \), or equivalently,

\[ u(x,y) = f(x-3y) \cos(3x+y) + g(x-3y) \sin(3x+y) \]

is the general solution of (a) with \( f \) and \( g \) arbitrary \( C^2 \)-functions of a single real variable.

16 pts. To solve the p.d.e. in (c) note that it is equivalent to

\[ 0 = (\frac{\partial}{\partial x} - 3 \frac{\partial}{\partial y})(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})u + 2(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}) \]

Let \( \{ \begin{align*}
\xi &= 3x + y \\
\eta &= x + y
\end{align*} \)

Then the chain rule implies, as operators we have

18 pts. To solve the p.d.e. in (c) is equivalent to

\[ \Rightarrow \frac{\partial^2 u}{\partial \eta^2} + u = 0 \]

(over)
\[
\frac{2}{dx} = \frac{d}{dx} \frac{d}{dz} + \frac{dy}{dx} \frac{d}{dy} = \frac{d^2}{dz} + \frac{2}{\eta} \quad \text{and} \quad \frac{2}{dy} = \frac{d}{dy} \frac{d}{dz} + \frac{dx}{dy} \frac{d}{dx} = \frac{2}{\eta} + \frac{2}{\eta}.
\]

Thus \[ \frac{2}{dx} - 3 \frac{2}{dy} = 3 \frac{2}{\eta} + \frac{2}{\eta} - 3 \left( \frac{2}{\eta} + \frac{2}{\eta} \right) = -2 \frac{2}{\eta} \]

and \[ \frac{2}{dy} - 2 \frac{2}{dy} = 3 \frac{2}{\eta} + \frac{2}{\eta} - \left( \frac{2}{\eta} + \frac{2}{\eta} \right) = 2 \frac{2}{\eta} . \]

Consequently, the p.d.e. in (c) is equivalent to
\[ (-2 \frac{2}{\eta}) (2 \frac{2}{\eta}) u + 2 (2 \frac{2}{\eta}) u = 0 \]

\[ -\eta \left( \frac{du}{dz} \right) + \frac{du}{dz} = 0 . \]

Letting \( v = \frac{du}{dz} \), this becomes \( \frac{dv}{\eta} - v = 0 \). Therefore, \( v = c(s)e^\eta \).

That is, \( \frac{du}{dz} = c(s)e^\eta \) so \( u = \int c(s)e^\eta dz = e^\eta \int c(s)dz = e^\eta (s(s) + c(y)) \)

Consequently, \( u = f(s)e^\eta + g(y) \) \( (g(y) = e^\eta c(y)) \),

or equivalently
\[ u(x,y) = f(3x+y)e^{x+y} + g(x+y) \]

is the general solution of (c) where f and g are arbitrary \( C^2 \)-functions of a single real variable.
4. (25 pts.) Let \( u = u(x,t) \) be a nonconstant \( C^3 \) solution to
\[ u_{tt} - u_{xx} + u_t = 0 \]
in the upper half-plane \(-\infty < x < \infty, \ 0 \leq t < \infty\), such that for each fixed \( t \geq 0 \), \( u_t, u_x, u_{xx}, u_{tt} \), and \( u_t \) are square-integrable functions of \( x \) on \((-\infty, \infty)\) and \( \lim_{|x| \to \infty} u_t(x,t) = 0 \).

(a) Show that \( G(t) = \int_{-\infty}^{\infty} \left\{ \left[ u_t(x,t) \right]^2 + \left[ u_x(x,t) \right]^2 \right\} \, dx \) is a decreasing function for \( t \geq 0 \).

(b) Give a physical interpretation of the result in part (a).

\[ G'(t) = \frac{d}{dt} \int_{-\infty}^{\infty} \left\{ \left[ u_t(x,t) \right]^2 + \left[ u_x(x,t) \right]^2 \right\} \, dx \]
\[ = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left\{ \left[ u_t(x,t) \right]^2 + \left[ u_x(x,t) \right]^2 \right\} \, dx \]
\[ = 2 \int_{-\infty}^{\infty} \left[ u_t(x,t) u_{tt}(x,t) + u_x(x,t) u_{tx}(x,t) \right] \, dx \]

But \( \int_{-\infty}^{\infty} u_x(x,t) u_{tx}(x,t) \, dx = \lim_{M \to \infty} \int_{-\infty}^{N} \frac{u_t(x,t) u_x(x,t) \, dx}{dx} \) (Integrate by parts)
\[ = \lim_{M \to \infty} \left( u_x(x,t) u_t(x,t) \right)_{x=N}^{x=M} - \int_{N}^{M} u_t(x,t) u_{xx}(x,t) \, dx \]
\[ = -\int_{-\infty}^{\infty} u_t(x,t) u_{xx}(x,t) \, dx. \]

Therefore, \( G'(t) = 2 \int_{-\infty}^{\infty} \left[ u_t(x,t) u_{tt}(x,t) - u_t(x,t) u_{xx}(x,t) \right] \, dx \)
\[ = 2 \int_{-\infty}^{\infty} u_t(x,t) \left[ u_{tt}(x,t) - u_{xx}(x,t) \right] \, dx \]
\[ = -2 \int_{-\infty}^{\infty} u_t(x,t) u_t(x,t) \, dx \quad \text{(since } u \text{ solves } u_{tt} - u_{xx} + u = 0) \]

(over)
so \( G'(t) = -2 \int_{-\infty}^{\infty} [u_t(x,t)]^2 \, dx \leq 0 \) and hence \( G \) is a decreasing function (in the weak sense) on \( t \geq 0 \).

(b) The equation \( u_{tt} - u_{xx} + u_t = 0 \) models the small vibrations of a homogeneous elastic string with tension \( T_0 = 1 \), density \( \rho_0 = 1 \), and damping constant \( r = 1 \). (See problem #2 on this exam.) The energy at time \( t \) of a solution \( u = u(x,t) \) to \( u_{tt} - u_{xx} + u_t = 0 \) is

\[
E(t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \rho_0 u_t^2(x,t) + \frac{1}{2} T_0 u_x^2(x,t) \right] \, dx = \frac{1}{2} G(t).
\]

Since damping transfers energy from the vibrating string to the environment, one expects \( E(t) \) (and consequently \( G(t) \)) to decrease with time \( t \).