Mathematics 325 | Exam II Fall 2012 | Name: Dr. Grow

1.(33 pts.) Solve $u_t - Ku_{xx} = 0$ in the upper half-plane $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = x^3$ if $-\infty < x < \infty$. You may find the fact $\int_{-\infty}^{\infty} e^{-p^2} p^2\,dp = \frac{\sqrt{\pi}}{2}$ useful.

If $\varphi = \varphi(x)$ is continuous and bounded on $(-\infty,\infty)$ then $u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$ (2) solves $u_t - ku_{xx} = 0$ in the upper half-plane and satisfies the initial condition $u(x,0) = \varphi(x)$ if $-\infty<x<\infty$ in the sense that $\lim_{t\to 0^+} u(x,t) = \varphi(x)$ for $-\infty<x<\infty$.

Although $\varphi(x) = x^3$ is not bounded on $(-\infty,\infty)$, we apply the solution formula and will check our answer at the end.

(3) $u(x,t) = \frac{1}{\sqrt{4K\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Kt}} y^3\,dy$. Let $p = \frac{y-x}{\sqrt{4Kt}}$ (2). Then $dp = \frac{dy}{\sqrt{4Kt}}$ (2) so

(3) $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x + p\sqrt{4Kt})^3\,dp = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\left(x^3 + 3x^2 p\sqrt{4Kt} + 3x(p\sqrt{4Kt})^2 + (p\sqrt{4Kt})^3\right)dp$ (3)

(3) $= \frac{x^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp + \frac{3x^2\sqrt{4Kt}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p\,dp + \frac{12Ktx}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp + \frac{4Kt\sqrt{4Kt}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^3\,dp.$

But $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp = 1$, $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp = \frac{1}{2}$, and oddness of the integrands imply

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p\,dp = 0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} p^3\,dp$. (3) (3) Thus $u(x,t) = \boxed{x^3 + 6Ktx}$. (6)

Check: $u_t - Ku_{xx} = 6Kx - K(6x) \overset{\checkmark}{=} 0$ for all $-\infty<x<\infty$, $-\infty<t<\infty$. (3)

$u(x,0) \overset{\checkmark}{=} x^3$ if $-\infty<x<\infty$.

2.(33 pts.) (a) Under appropriate hypotheses on a function $g = g(x)$ defined on $-\infty < x < \infty$, show that

$$\mathfrak{F}(g(x-a))(\xi) = \hat{g}(\xi)e^{-i\xi a} \text{ and } \mathfrak{F}\left(\int_{-\infty}^{x} g(s)\,ds\right)(\xi) = \frac{\hat{g}(\xi)}{i\xi}.$$

(b) Use Fourier transform methods to derive a formula for the solution to

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad -\infty < x < \infty, \ -\infty < t < \infty, \quad (1)$$

subject to $u(x,0) = \phi(x)$ (2) and $u_t(x,0) = \psi(x)$ (3) if $-\infty < x < \infty$. You may find it useful to recall the identities

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

(a) $\mathfrak{F}(g(\cdot - a))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-a)e^{-i\xi x}\,dx$. Let $z = x-a$. Then $dz = dx$ so

2 $$\mathfrak{F}(g(\cdot - a))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(z)e^{-i\xi(z+a)}\,dz = \frac{e^{-i\xi a}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(z)e^{-i\xi z}\,dz = e^{-i\xi a}\hat{g}(\xi).$$

(This formula holds for all absolutely integrable functions g on $(-\infty,\infty)$.)

Let $f(x) = \int_{-\infty}^{x} g(s)\,ds$ and $f'(x) = g(x)$ be absolutely integrable on $(-\infty,\infty)$.

Since f and f' are absolutely integrable on $(-\infty,\infty)$, an identity proved in lecture shows that $\mathfrak{F}(f')(\xi) = i\xi\hat{f}(\xi)$ for all real ξ. That is,

2 $$\frac{\hat{g}(\xi)}{i\xi} = \frac{\mathfrak{F}(f')(\xi)}{i\xi} = \hat{f}(\xi) = \mathfrak{F}\left(\int_{-\infty}^{\cdot} g(s)\,ds\right)(\xi) \quad \text{for all real } \xi \neq 0.$$

(b) Let $u = u(x,t)$ be a solution to (1)-(2)-(3). Then taking the Fourier transform of (1) with respect to x gives

3 $$\frac{\partial^2}{\partial t^2}\mathfrak{F}(u)(\xi) + c^2\xi^2\mathfrak{F}(u)(\xi) = \mathfrak{F}(u_{tt})(\xi) - c^2(i\xi)^2\mathfrak{F}(u)(\xi) = \mathfrak{F}(u_{tt} - c^2u_{xx})(\xi) = \mathfrak{F}(0)(\xi) = 0.$$

The general solution of this ODE in t (with parameter ξ) is

3 (*) $$\mathfrak{F}(u)(\xi) = k_1(\xi)\cos(c\xi t) + k_2(\xi)\sin(c\xi t).$$

1 Applying (2) yields $\hat{\varphi}(\xi) = \mathfrak{F}(u(\cdot,0))(\xi) = k_1(\xi)$. On the other hand,

1 $$\mathfrak{F}(u_t)(\xi) = \frac{\partial}{\partial t}\mathfrak{F}(u)(\xi) = -c\xi k_1(\xi)\sin(c\xi t) + c\xi k_2(\xi)\cos(c\xi t)$$

(1) so applying ③ gives $\hat{\psi}(\xi) = \mathcal{F}(u_t(\cdot,0))(\xi) = c\xi\, k_2(\xi)$. Substituting these expressions for k_1 and k_2 into (*) and using the given identities produces

(3)
$$\mathcal{F}(u)(\xi) = \hat{\varphi}(\xi)\cos(c\xi t) + \frac{\hat{\psi}(\xi)}{c\xi}\sin(c\xi t) = \hat{\varphi}(\xi)\left(\frac{e^{ic\xi t}+e^{-ic\xi t}}{2}\right) + \frac{\hat{\psi}(\xi)}{c\xi}\left(\frac{e^{ic\xi t}-e^{-ic\xi t}}{2i}\right)$$

Applying the results of part (a) of this problem lead to

(6)
$$\mathcal{F}(u)(\xi) = \frac{1}{2}\mathcal{F}(\varphi(\cdot+ct))(\xi) + \frac{1}{2}\mathcal{F}(\varphi(\cdot-ct))(\xi) + \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{\cdot+ct}\psi(s)\,ds\right)(\xi) - \frac{1}{2c}\mathcal{F}\left(\int_{-\infty}^{\cdot-ct}\psi(s)\,ds\right)(\xi)$$

(6)
$$= \mathcal{F}\left(\frac{1}{2}\left[\varphi(\cdot+ct)+\varphi(\cdot-ct)\right] + \frac{1}{2c}\int_{\cdot-ct}^{\cdot+ct}\psi(s)\,ds\right)(\xi).$$

The inversion theorem then implies

(3)
$$\boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+ct)+\varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds}$$

for $-\infty<x<\infty$ and each real number t.

3.(34 pts.) Solve the beam equation

$$u_{tt} + u_{xxxx} \overset{①}{=} 0 \text{ if } 0 < x < 1,\ 0 < t < \infty,$$

subject to the boundary conditions

②-③-④-⑤ $$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \quad \text{if } t \geq 0,$$

and the initial conditions

$$u(x,0) \overset{⑥}{=} 2\sin(\pi x) - 3\sin(5\pi x) \quad \text{and} \quad u_t(x,0) \overset{⑦}{=} 0 \quad \text{if } 0 \leq x \leq 1.$$

You may find useful the fact that the general solution to $y^{(4)} - \beta^4 y = 0$ is

$$y(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x) + c_3 \cos(\beta x) + c_4 \sin(\beta x).$$

We use the method of separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem, ①-②-③-④-⑤-⑦, of the form $u(x,t) = X(x)T(t)$. Substituting in ① gives

$$X(x)T''(t) + X^{(4)}(x)T(t) = 0 \quad \text{or} \quad \frac{-T''(t)}{T(t)} = \frac{X^{(4)}(x)}{X(x)} = \text{constant} = \lambda.$$ 2

Substituting in ② gives $X(0)T(t) = 0$ if $t \geq 0$ and since $u(x,t) = X(x)T(t)$ is not identically zero on $0 \leq x \leq 1$, $0 \leq t < \infty$, it follows that $X(0) = 0$. Similar arguments using ③-④-⑤-⑦ leads to a coupled system of ODEs and BCs:

$$\begin{cases} X^{(4)}(x) - \lambda X(x) \overset{⑧}{=} 0, & X(0) \overset{⑨}{=} 0 \overset{⑩}{=} X(1),\ X''(0) \overset{⑪}{=} 0 \overset{⑫}{=} X''(1), \\ T''(t) + \lambda T(t) \overset{⑬}{=} 0, & T'(0) \overset{⑭}{=} 0. \end{cases}$$ 6 4

We will assume that the eigenvalues λ of the eigenvalue problem ⑧-⑨-⑩-⑪-⑫ are real. (See Chapter 5 for the justification of this assumption.) We first claim that the eigenvalues λ are nonnegative. To see this, suppose that λ is an eigenvalue with corresponding eigenfunction $X = X(x)$. Then integrating by parts twice shows that

$$\lambda \int_0^1 X^2(x)\,dx = \int_0^1 X(x)(\lambda X(x))\,dx = \int_0^1 X(x) X^{(4)}(x)\,dx = \left(X(x)X^{(3)}(x) - X'(x)X''(x)\right)\Big|_0^1 + \int_0^1 (X''(x))^2\,dx$$

Now ⑨-⑩-⑪-⑫ imply that $\left(X(x)X^{(3)}(x) - X'(x)X''(x)\right)\Big|_0^1 = 0$ so

$$\lambda \int_0^1 X^2(x)\,dx = \int_0^1 (X''(x))^2\,dx \geq 0.$$

But $\int_0^1 X^2(x)\,dx > 0$ since $X = X(x)$ is not identically zero, and hence $\lambda \geq 0$ as claimed. 4

<u>Case $\lambda > 0$</u>, say $\lambda = \beta^4$ where $\beta > 0$.

Then ⑧ becomes $X^{(4)}(x) - \beta^4 X(x) = 0$ and this has general solution $X(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x) + c_3\cos(\beta x) + c_4\sin(\beta x)$ with $X''(x) = \beta^2 c_1\cosh(\beta x) + \beta^2 c_2\sinh(\beta x) - \beta^2 c_3\cos(\beta x) - \beta^2 c_4\sin(\beta x)$. ⑨ and ⑪ yield $0 = X(0) = c_1 + c_3$ and $0 = X''(0) = \beta^2 c_1 - \beta^2 c_3$ from which $c_1 = 0 = c_3$ follows. Applying ⑩ and ⑫ leads to $0 = X(1) = c_2\sinh(\beta) + c_4\sin(\beta)$ and $0 = X''(1) = \beta^2 c_2\sinh(\beta) - \beta^2 c_4\sin(\beta)$. Summing the equations gives $2c_2\sinh(\beta) = 0$ and hence $c_2 = 0$ since $\sinh(\beta) > 0$ for $\beta > 0$. Substituting then gives $c_4\sin(\beta) = 0$. The eigenvalue condition – i.e. the condition for the existence of a nontrivial solution – is $\sin(\beta) = 0$. Consequently the positive eigenvalues and corresponding eigenfunctions are

(4) ⑮ $$\lambda_n = \beta_n^4 = (n\pi)^4 \quad \text{and} \quad X_n(x) = \sin(\beta_n x) = \sin(n\pi x) \qquad (n = 1, 2, 3, \dots).$$

<u>Case $\lambda = 0$</u>. Then ⑧ becomes $X^{(4)}(x) = 0$ with general solution $X(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ and $X''(x) = 2c_2 + 6c_3 x$. Applying ⑨ and ⑪ yield $0 = X(0) = c_0$ and $0 = X''(0) = 2c_2$. Applying ⑫ and ⑩ then give $0 = X''(1) = 2c_2 + 6c_3$ (so $c_3 = 0$) and $0 = X(1) = c_1 + c_1 + c_2 + c_3$ (so $c_1 = 0$). Hence there are no nontrivial solutions in this case; i.e.

(2) zero is not an eigenvalue.

(2) Substituting from ⑮ into ⑬-⑭ and solving easily gives $T_n(t) = \cos(n^2\pi^2 t)$, up to a constant factor. The superposition principle yields that

(4) $$u(x,t) = \sum_{n=1}^{N} c_n X_n(x) T_n(t) = \sum_{n=1}^{N} c_n \sin(n\pi x)\cos(n^2\pi^2 t)$$

satisfies ①-②-③-④-⑤-⑦ for any integer $N \geq 1$ and any constants $c_1, c_2, \dots, c_N$. We need to choose N and the c_ns so that ⑥ is satisfied:

(2) $$2\sin(\pi x) - 3\sin(5\pi x) = u(x,0) = \sum_{n=1}^{N} c_n \sin(n\pi x) \quad \text{for all } 0 \leq x \leq 1.$$

(2) By inspection, we may take $N = 5$, $c_1 = 2$, $c_5 = -3$, and all other $c_n = 0$. That is,

(2) $$\boxed{u(x,t) = 2\sin(\pi x)\cos(\pi^2 t) - 3\sin(5\pi x)\cos(25\pi^2 t)}$$

is a solution of ①-②-③-④-⑤-⑥-⑦. (Note: Energy techniques can be used to show that this is the unique solution of the problem.)

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx$
A.	$\begin{cases} 1 & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0<x\le b, \\ 2b-x & \text{if } b<x<2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x>0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b<x<c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases} e^{iax} & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert\ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a. \end{cases}$

Math 325

Exam II

Fall 2012

$n = 30$

mean = 66.9

median = 68

standard deviation = 12.8

Range	Graduate Grade	Undergraduate Grade	Frequency
87 – 100	A	A	1
73 – 86	B	B	8
60 – 72	C	B	16
50 – 59	C	C	2
0 – 49	F	D	3