Mathematics 325 Exam II Summer 2009 Name: Dr. Grow

1.(25 pts.) (a) Let $u = u(x,t)$ be a classical solution to the diffusion equation $u_t - u_{xx} \overset{1}{=} 0$ in the strip $0 < x < 1$, $0 < t < \infty$, such that $u(0,t) \overset{2}{=} 0 \overset{3}{=} u_x(1,t)$ if $t \geq 0$. Show that the energy of the solution u, given by $E(t) = \int_0^1 u^2(x,t)\,dx$, is a nonincreasing function on the interval $0 \leq t < \infty$.

8 pts. to here.

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^1 u^2(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t}\left[u^2(x,t)\right]dx = \int_0^1 2u(x,t)u_t(x,t)\,dx \overset{\text{by } 1}{=} \int_0^1 \overbrace{2u(x,t)}^{v}\,\overbrace{u_{xx}(x,t)\,dx}^{dV}$$

$$= 2u(x,t)u_x(x,t)\Big|_{x=0}^{1} - \int_0^1 2u_x(x,t)u_x(x,t)\,dx = 2u(1,t)\underbrace{\cancel{u_x(1,t)}}_{0 \text{ by } 3} - 2\underbrace{\cancel{u(0,t)}}_{0 \text{ by } 2}u_x(0,t) - 2\int_0^1 u_x^2(x,t)\,dx$$

14 pts. to here.

$$= -2\int_0^1 u_x^2(x,t)\,dx \leq 0.$$

15 pts. to here.

Therefore $E = E(t)$ is a nonincreasing function on the interval $0 \leq t < \infty$.

(b) Use part (a) to help show that there is at most one classical solution to the following problem.

$$(*)\begin{cases} u_t - u_{xx} = f(x,t) \text{ if } 0 < x < 1,\ 0 < t < \infty, \\ u(0,t) = g(t) \text{ and } u_x(1,t) = h(t) \text{ if } 0 \leq t < \infty, \\ u(x,0) = \phi(x) \text{ if } 0 \leq x \leq 1. \end{cases}$$

(Here $f, g, h,$ and ϕ are fixed continuous functions.)

2 pts. to here.

Suppose $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ are classical solutions to $(*)$. Then $v(x,t) = u_1(x,t) - u_2(x,t)$ is a classical solution to

4 pts. to here.

$$(**)\begin{cases} v_t - v_{xx} = 0 \text{ if } 0 < x < 1,\ 0 < t < \infty, \\ v(0,t) = 0 = v_x(1,t) \text{ if } 0 \leq t < \infty, \\ v(x,0) = 0 \text{ if } 0 \leq x \leq 1. \end{cases}$$

6 pts. to

8 pts. to here.

If $E(t) = \int_0^1 v^2(x,t)\,dx$ $(t \geq 0)$ is the energy of the solution v, then E is a nonincreasing function on the interval $[0,\infty)$. But $0 \leq E(t) \leq E(0) = \int_0^1 v^2(x,0)\,dx = 0$ for all $t \geq 0$ so by the vanishing theorem, the nonnegative continuous integrand of E must be identically zero for each $t \geq 0$ and $0 \leq x \leq 1$. I.e. $v(x,t) \equiv 0$ so $\boxed{u_1 = u_2}$.

10 pts. to here.

2.(25 pts.) Solve the heat equation in the upper half-plane

$$u_t - u_{xx} = 0 \text{ if } -\infty < x < \infty,\ 0 < t < \infty,$$

subject to the initial condition

$$u(x,0) = x^2 \text{ if } -\infty < x < \infty.$$

Note: You may find the following facts useful.

$$\int_{-\infty}^{\infty} pe^{-p^2}\,dp = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}.$$

Although $\varphi(x) = x^2$ is not bounded on $-\infty < x < \infty$, we use the formula for solutions to the diffusion I.V.P. in the upper half plane:

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy \qquad (t>0).$$

4 pts. to here.

Since $k=1$ and $\varphi(x) = x^2$ this becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^2\,dy \quad \cdots\cdots \quad \text{Let } p = \frac{y-x}{\sqrt{4t}}. \text{ Then } dp = \frac{dy}{\sqrt{4t}}.$$

12 pts. to here.

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left(x + p\sqrt{4t}\right)^2 dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \left(x^2 + 2px\sqrt{4t} + 4tp^2\right) dp$$

$$= \frac{x^2}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2}\,dp}_{\sqrt{\pi}} + \frac{2x\sqrt{4t}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} pe^{-p^2}\,dp}_{0} + \frac{4t}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp}_{\frac{\sqrt{\pi}}{2}}$$

$$\boxed{u(x,t) = x^2 + 2t}$$

23 pts. to here.

Since $\varphi(x) = x^2$ is not bounded on $-\infty < x < \infty$, we have "abused" the formula for solutions to the diffusion I.V.P. in the upper halfplane and must check our answer.

$$u_t - u_{xx} = 2 - 2 = 0 \ \checkmark$$

$$u(x,0) = x^2 \ \checkmark$$

25 pts. to here.

3.(25 pts.) Use the Fourier transform method to derive a formula for the solution to the following initial value problem.

$$u_t - u_{xx} + tu \overset{①}{=} 0 \text{ if } -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) \overset{②}{=} f(x) \text{ if } -\infty < x < \infty.$$

Taking the Fourier transform of ① with respect to x yields $\mathcal{F}(u_t - u_{xx} + tu)(\xi) = \mathcal{F}(0)(\xi)$

so $\mathcal{F}(u_t)(\xi) - \mathcal{F}(u_{xx})(\xi) + t\mathcal{F}(u)(\xi) = 0$ by linearity. But $\mathcal{F}(f')(\xi) = i\xi\mathcal{F}(f)$

2 pts. to here.

and $\mathcal{F}(u_t)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t) e^{-i\xi x}\,dx = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx\right) = \frac{\partial}{\partial t}\mathcal{F}(u)(\xi)$ so

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + t\mathcal{F}(u)(\xi) = 0$$

5 pts. to here.

$$(\dagger) \qquad \frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (t+\xi^2)\mathcal{F}(u)(\xi) = 0.$$

6 pts. to here

An integrating factor for this linear first-order ODE in the independent variable t (and parameter ξ) is $\mu(t) = e^{\int (t+\xi^2)dt} = e^{\frac{t^2}{2}+\xi^2 t}$. Multiplying ($\dagger$) by the integrating factor,

to here.

we obtain an "exact" expression in the left member:

$$\frac{\partial}{\partial t}\left(e^{\frac{t^2}{2}+\xi^2 t}\mathcal{F}(u)(\xi)\right) = e^{\frac{t^2}{2}+\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (t+\xi^2)e^{\frac{t^2}{2}+\xi^2 t}\mathcal{F}(u)(\xi) = 0.$$

Integrating with respect to t (holding ξ fixed) yields

$$e^{\frac{t^2}{2}+\xi^2 t}\mathcal{F}(u)(\xi) = c(\xi) \quad \Rightarrow \quad \mathcal{F}(u)(\xi) = c(\xi)e^{-\frac{t^2}{2}-\xi^2 t}$$

12 pts. to here.

Applying the initial condition ② yields $\mathcal{F}(f)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c(\xi)$ so

14 pts. to here.

(*) $\mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)e^{-\frac{t^2}{2}} \cdot e^{-\xi^2 t}$. Applying formula I in the table of Fourier transforms: $\mathcal{F}(e^{-ax^2})(\xi) = \frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}$ with $\frac{1}{4a} = t$, we find

$\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{x^2}{4t}}\right)(\xi) = \mathcal{F}\left(\sqrt{2a}\,e^{-ax^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}} = e^{-\xi^2 t}$. Substituting

16 pts. to here.

into (*) we have $\mathcal{F}(u)(\xi) = e^{-\frac{t^2}{2}}\mathcal{F}(f)(\xi)\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{x^2}{4t}}\right)(\xi)$. (OVER)

But $\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$ so the last expression is equivalent to

20 pts. to here.

$$\mathcal{F}(u)(\xi) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}\mathcal{F}\left(f * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi) = \mathcal{F}\left(f * \frac{1}{\sqrt{4\pi t}}e^{-\frac{t^2}{2}-\frac{(\cdot)^2}{4t}}\right)(\xi).$$

By the uniqueness theorem for Fourier transforms, it follows that

22 pts. to here

$$u(x,t) = \left(f * \frac{1}{\sqrt{4\pi t}}e^{-\frac{t^2}{2}-\frac{(\cdot)^2}{4t}}\right)(x)$$

or equivalently

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}-\frac{(x-y)^2}{4t}} f(y)\,dy$$

25 pts. to here.

$$\boxed{u(x,t) = \frac{e^{-t^2/2}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy}$$

for $t>0$ and $-\infty<x<\infty$.

4.(25 pts.) Solve the diffusion problem in the strip

$$u_t - u_{xx} \overset{(1)}{=} 0 \text{ if } 0 < x < 1,\ 0 < t < \infty,$$

subject to the mixed boundary conditions (Dirichlet at the left end, Neumann at the right end)

$$u(0,t) \overset{(2)}{=} 0 \overset{(3)}{=} u_x(1,t) \text{ if } 0 \le t < \infty,$$

and the initial condition

$$u(x,0) \overset{(4)}{=} 3\sin\left(\frac{\pi x}{2}\right) - \sin\left(\frac{3\pi x}{2}\right) \text{ if } 0 \le x \le 1.$$

(Note: The solution to this problem is unique by problem 1 on this exam.)

We use separation of variables. We seek nontrivial solutions of (1)-(2)-(3) of the form

5 pts. to here. $u(x,t) = X(x)T(t)$. Substituting into (1) and rearranging we have $-\frac{T'(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \lambda$

for some constant λ and all $0<x<1$, $0<t<\infty$. Substituting into (2) and (3) yields

$X(0)T(t) = 0$ and $X'(1)T(t) = 0$ for all $t \ge 0$. Since we require that $T(t) \not\equiv 0$

7 pts. to here. it follows that $X(0) = 0 = X'(1)$. Therefore we are led to the coupled system of ODE's

$$\begin{cases} \boxed{X''(x) + \lambda X(x) = 0 \text{ if } 0<x<1 \text{ with } X(0) = 0 = X'(1)}; \\ \text{and} \\ T'(t) + \lambda T(t) = 0 \text{ if } t > 0. \end{cases}$$

Eigenvalue Problem

We assume that the eigenvalues λ above are real.

Case $\lambda = 0$: Then $X''(x) = 0$ has general solution $X(x) = c_1 x + c_2$. $0 = X(0) = c_2$

11 and $0 = X'(1) = c_1$ so there are no nontrivial solutions in this case; i.e. $\lambda = 0$ is not an eigenvalue.

Case $\lambda < 0$ (say $\lambda = -\mu^2$ where $\mu > 0$): Then $X''(x) - \mu^2 X(x) = 0$ has general solution $X(x) = c_1\cosh(\mu x) + c_2\sinh(\mu x)$. Then $0 = X(0) = c_1$ and $0 = X'(1) = \mu c_2\cosh(\mu)$

13 pts. to here. imply $c_1 = c_2 = 0$. I.e. there are no nontrivial solutions in this case and hence there are no negative eigenvalues.

Case $\lambda > 0$ (say $\lambda = \mu^2$ where $\mu > 0$): Then $X''(x) + \mu^2 X(x) = 0$ has general solution $X(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$. Also $0 = X(0) = c_1$ and $0 = X'(1) = \mu c_2\cos(\mu)$

(OVER)

imply either $c_1 = c_2 = 0$ or $c_1 = 0$ and $\cos(\mu) = 0$. Therefore in order to have a nontrivial solution in this case, we must have $\mu_n = (n+\frac{1}{2})\pi$ for some integer $n = 0, 1, 2, \ldots$ Hence the eigenvalues are

15 pts. to here.
$$\lambda_n = \mu_n^2 = (n+\tfrac{1}{2})^2\pi^2 \qquad (n = 0,1,2,\ldots)$$

and the corresponding eigenfunctions are (up to a constant factor)

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$$X_n(x) = \sin(\mu_n x) = \sin((n+\tfrac{1}{2})\pi x) \qquad (n=0,1,2,\ldots)$$

The corresponding solutions to the t-equation,

$$T_n'(t) + \lambda_n T_n(t) = 0 \quad \text{or equivalently} \quad T_n'(t) + (n+\tfrac{1}{2})^2\pi^2 T_n(t) = 0,$$

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are (up to a constant factor) $T_n(t) = e^{-(n+\frac{1}{2})^2\pi^2 t}$. Therefore

$$u_n(x,t) = X_n(x)T_n(t) = e^{-(n+\frac{1}{2})^2\pi^2 t}\sin((n+\tfrac{1}{2})\pi x) \qquad (n=0,1,2,\ldots)$$

are nontrivial solutions to ①-②-③. By the superposition principle

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$$u(x,t) = \sum_{n=0}^{N} b_n e^{-(n+\frac{1}{2})^2\pi^2 t}\sin((n+\tfrac{1}{2})\pi x)$$

also solves ①-②-③ for any integer $N \geq 0$ and any choice of constants $b_0, b_1, \ldots, b_N$. In order to satisfy ④ as well, we must have

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$$3\sin\left(\frac{\pi x}{2}\right) - \sin\left(\frac{3\pi x}{2}\right) \overset{④}{=} u(x,0) = \sum_{n=0}^{N} b_n \sin((n+\tfrac{1}{2})\pi x) \quad \text{for all } 0 \leq x \leq 1.$$

By inspection, we may take $N = 1$ and $b_0 = 3$, $b_1 = -1$ to satisfy ④. Thus

25 pts. to here.
$$\boxed{u(x,t) = 3e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi x}{2}\right) - e^{-\frac{9\pi^2}{4}t}\sin\left(\frac{3\pi x}{2}\right)}$$

solves ①-②-③-④.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}} \dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}} \dfrac{e^{-a\|\xi\|}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\dfrac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}} \dfrac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}} \dfrac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \|\xi\| \ge a, \\ \sqrt{\pi/2} & \text{if } \|\xi\| < a. \end{cases}$

Math 325

Exam II

Summer 2009

mean: 78.1

standard deviation: 17.6

number of scores: 16

Distribution of Scores:

		frequency
87 – 100	A	6
73 – 86	B	4
60 – 72	C (graduate), B (undergraduate)	4
50 – 59	C	1
0 – 49	F (graduate), D (undergraduate)	1